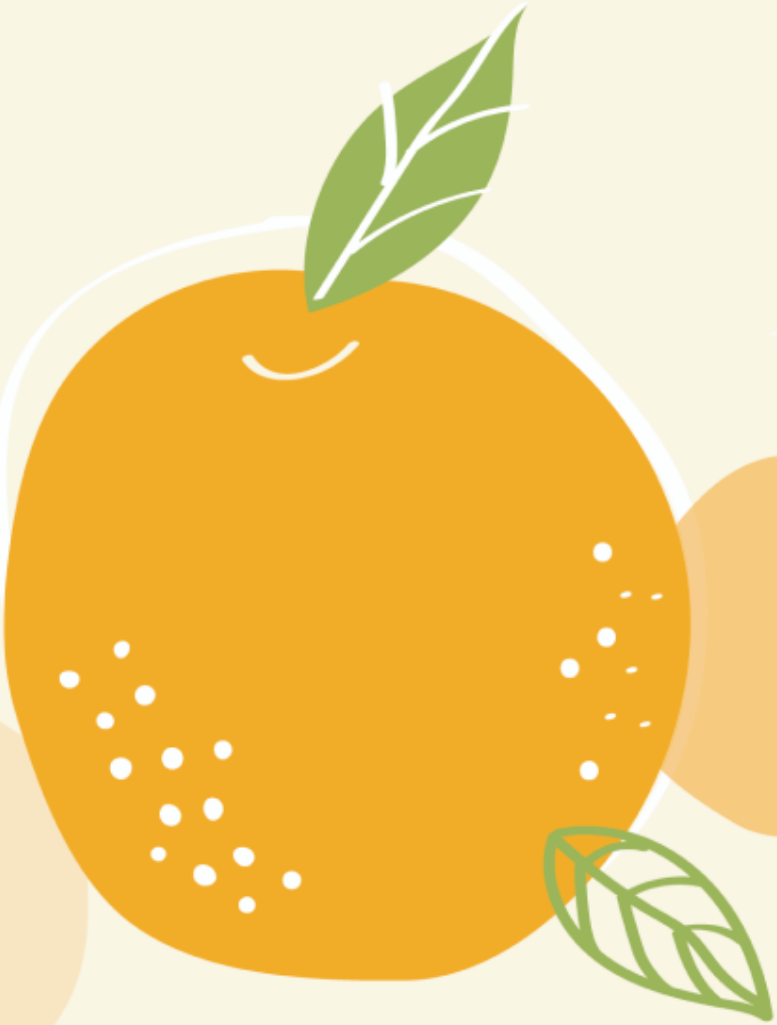
MA 408

measure theory



No attendance
4 quizes, 2 suprise
10% each quiz,
best 3/4 quiz
quiz - 30%.
midsem - 30%.
endsem - 40%.
Ref - upto extension of measures measure theory by Rana
Rest of course folland and royden
Extra sauerwise - measure theory part extra
Extra measure integration book for problems
Office hours - friday 3 to 5

5th Jan :

motivation for measure theory comes from → seq of fuction using pointwise does not garuntee riemann integrable
Bad interplay of pointwise convergence and Riemann integration

Eg: $\mathbb{Q} \cap [0,1] = \{q_j\}_{j=1}^{\infty}$

$\{f_n\}_{n=1}^{\infty}$ be s.t $f_n(x) = \begin{cases} 1 ; x \in \{a_i\}_{i=1}^{n} \\ 0 ; \text{otherwise} \end{cases}$

then $f_n \to f$ pointwise where $f(x) = \begin{cases} 1 ; x \in \mathbb{Q} \cap [0,1] \\ 0 ; \text{otherwise} \end{cases}$

Note: $f_n : [0,1] \to \mathbb{R}$ is riemann integrable but $f : [0,1] \to \mathbb{R}$ is not Riemann integrable

for reasons like this we want a new integration theory so it behaves nicely with pointwise convergence

measure of interval is length of the interval, we want to define measure for a more complex set

Eg: 1D - length
2D - Area
3D - volume measure refers to volume

Remedy is to generalise notion of "length" in the definition of the Riemann integration

properties of such functions :

$\forall a, b \in \mathbb{R}$, s.t $b > a$:

1. $\lambda((a,b)) = \lambda((a,b]) = \lambda([a,b)) = \lambda([a,b]) = b - a$

we should also be able to measure $\mathbb{R}$

2. $\lambda((-\infty, a)) = \lambda((-\infty, a]) = \infty$
3. $\lambda((a, \infty)) = \lambda([a, \infty)) = \infty$
4. $\lambda((-\infty, \infty)) = \infty$

now, let I be the set of all intervals of the above form

Defn: $\lambda : I \to [0, \infty]$ is called a length function on I

Let $\mathcal{P}(\mathbb{R})$ be power set of $\mathbb{R}$
suppose $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ be an extension of λ, then μ will have

1. $\mu(I) = \lambda(I) \quad \forall I \in I$
2. $\mu(x + A) = \mu(A) \quad \forall A \in \mathcal{P}(\mathbb{R}), x \in \mathbb{R}$ → $\{x + a \mid a \in A\} = x + A$
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ for $\{A_i\}_{i=1}^{\infty} \subset \mathcal{P}(\mathbb{R})$ s.t $A_i \cap A_j = \phi \ \forall i, j$ → σ-additive or countable additivity

countable identity not uncountable summation does not make sense, more to start, more to find

1, 2, 3 ⇒ 4. $\mu(\phi) = 0$
as $A_1 = A$, $A_2 = \phi$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{2} A_i\right) = \mu(A) + \mu(\phi) = \mu(A)$$
$$\Rightarrow \mu(\phi) = 0$$

also, 1, 2, 3 ⇒ 5. $\mu(E) \leq \mu(F)$ if $E \subseteq F$

if $\mu(F) = \infty$ then S. holds trivially, otherwise

$\mu(F) = \mu(E \cup (F \setminus E))$

$= \mu(E) + \mu(F \setminus E)$

$\geq \mu(E)$ as $\mu(F \setminus E) \in [0, \infty)$ as $\mu(F \setminus E) \neq \infty$

Vitali's construction:

This tells us that we cannot have μ satisfying 1, 2, 3

define '$\sim$' on $\mathbb{R}$ by $x \sim y$ iff $x - y \in \mathbb{Q}$

$\sim$ is an equivalence relation (trivial)

let Λ denote set of equivalence classes under $\sim$

Λ is uncountable, as every equivalence class is $[x] = x + \mathbb{Q}$ is countable but $\mathbb{R}$ is uncountable

suppose $\alpha \in \Lambda$ then $\exists\, x_\alpha \in (0,1]$ s.t. $x_\alpha \in \alpha = [x_\alpha]$ (axiom of choice for x_α)

as $x - [x] \in [0,1)$ and $0 \sim 1$ (so we can shift x by $x - [x] \sim x$)

$\therefore \exists\, x_\alpha \in (0,1]$

define, $E = \{x_\alpha \in (0,1] \mid \alpha \in \Lambda\}$ (one x_α from every α) (only one)

then

I) $(p + E) \cap (q + E) = \phi \;\forall\, p, q \in \mathbb{Q}$ with $p \neq q$

II) $\bigcup_{q \in \mathbb{Q} \cap (-1,1)} (q + E) \subseteq (-1, 2)$

III) $(0,1) \subseteq \bigcup_{q \in \mathbb{Q} \cap (-1,1)} (q + E)$

I) suppose $p, q \in \mathbb{Q}$, $p \neq q$ and $(p + E) \cap (q + E) \neq \phi$

then $x_\alpha + p = x_\beta + q$ for some $\alpha, \beta \in \Lambda$

$\Rightarrow x_\alpha - x_\beta = q - p \in \mathbb{Q}$

$\Rightarrow x_\alpha \sim x_\beta$

$\Rightarrow \alpha = \beta$

$\Rightarrow x_\alpha = x_\beta$

$\Rightarrow p = q$

$p = q$ is a contradiction

II) Follows from definition of E

III) let $x \in (0,1)$, then $\exists\, \alpha \in \Lambda$ s.t. $x_\alpha \in (0,1]$ and $x \sim x_\alpha$

$\Rightarrow x - x_\alpha = q \in \mathbb{Q}$

$\Rightarrow q \in (-1,1) \cap \mathbb{Q}$

$\Rightarrow (0,1) \subseteq \bigcup_{q \in (-1,1) \cap \mathbb{Q}} (q + E)$

now, $\mu\left(\bigcup_{q \in \mathbb{Q} \cap (-1,1)} (q + E)\right) \leq \mu((-1,2)) = 3$ $(\because$ I, II$)$

$\Rightarrow \sum_{q \in \mathbb{Q} \cap (-1,1)} \mu(q + E) \leq 3$ $(\because 3)$

$\Rightarrow \sum_{q \in \mathbb{Q} \cap (-1,1)} \mu(E) \leq 3$ $(\because 2)$ (countable infinite $\mu(E)$)

$\Rightarrow \mu(E) = 0$

but from I, III, 3 $\mu((0,1)) \leq \sum_{q \in \mathbb{Q} \cap (-1,1)} \mu(q + E) = \sum_{q \in \mathbb{Q} \cap (-1,1)} \mu(E) = 0$

$\Rightarrow 1 \leq 0$ this is a contradiction

Note: From above, not all sets have length defined, so we start with sigma algebra

8th Jan:

Quiz-1 16th feb
Quiz-2 13th April

Notations:

① Any function $\mu: \mathcal{C} \to [0,\infty]$ where $\mathcal{C} \subseteq P(\Omega)$ for some $\Omega \neq \phi$ is called a set function

② given any subset $A \subseteq \Omega$, χ_A is the indicator fution of A it is also called characteristic fuction

③ $\mathbb{N}$ - set of positive integers $\{1, 2, \dots\}$

$\mathbb{N}_0$ - set of all non-negative integers $\{0, 1, \dots\}$

④ $(a,b] = \begin{cases} \phi ; & \text{if } b = a \text{ and } a \in \mathbb{R} \\ (a,\infty) ; & \text{if } b = \infty \\ \mathbb{R} ; & \text{if } a = -\infty, b = \infty \end{cases}$

⑤ Denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, extended + and · on $\overline{\mathbb{R}}$ partially

i) $x \pm \infty = \pm\infty \ \forall x \in \mathbb{R}$

ii) $x.(\pm\infty) = \pm\infty \ \forall x > 0$

iii) $x.(\pm\infty) = \mp\infty \ \forall x < 0$

iv) $\infty + \infty = \infty$

v) $-\infty - \infty = -\infty$

vi) $0.(\pm\infty) = 0$

this can give contradiction as $\lim_{x\to\infty} 5x+3 = \infty$, $\lim_{x\to\infty} 5x+1 = \infty$

Note: $\overline{\mathbb{R}}$ is partial extension as $\infty - \infty$ is forbidden

Class of sets and measures:

Let Ω be a non-empty set

Defn: $S \subseteq P(\Omega)$ is said to be a semi-algebra if

1) $\Omega \in S$
2) $\forall A, B \in S \Rightarrow A \cap B \in S$
3) given any $A \in S$, $\exists$ finitely many disjoint elements $E_1, \dots E_n$ in S

$$A^c = E_1 \cup \dots \cup E_n$$

Note: as $\Omega \in S$, $\Omega^c = \phi = \phi \in S$ from point 3

Eg: $\Omega = \mathbb{R}$

$$S = \{(a,b] \mid -\infty \leq a, b \leq \infty\}$$

$\mathbb{R} \in S$, and S is closed under taking intersection
moreover $(a,b]^c = (-\infty, a] \cup (b, \infty)$

then $(-\infty, a] \in S$, $(b, \infty) \in S$ and $(-\infty, a] \cap (b, \infty) = \phi$

Def: (Algebra) $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is called an Algebra if

1) $\Omega \in \mathcal{A}$
2) $\forall A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$
3) $\forall A \in \mathcal{A}$, $A^c \in \mathcal{A}$

Exe: An algebra $\mathcal{A}$ is also closed under finite unions → done down

Def: $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is called σ-algebra if

1) $\Omega \in \mathcal{F}$
2) $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$
3) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

Exe: A σ-algebra is closed under countable union → done down

Note: every σ-algebra is algebra and every algebra is semi-algebra

Exe: 1) let $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ be a collection of algebra on Ω
($\mathcal{A}_\lambda \subseteq \mathcal{P}(\Omega)$ is an algebra $\forall \lambda \in \Lambda$)

show that $\bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda \subseteq \mathcal{P}(\Omega)$ is an algebra

2) let $\{\mathcal{F}_\lambda \mid \lambda \in \Lambda\}$ be a collection of σ-algebra in Ω, show that $\bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda \subseteq \mathcal{P}(\Omega)$ is σ-algebra → done down

3) give counter example in case of semi algebra

Lemma: let $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ be a collection of subsets of Ω
then

$\mathcal{A}(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq A \subseteq \mathcal{P}(\Omega),\ A\text{-algebra}} A$ is an algebra containing $\mathcal{C}$

$\mathcal{F}_1(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq \mathcal{F}_1 \subseteq \mathcal{P}(\Omega),\ \mathcal{F}_1 \text{ is } \sigma\text{-algebra}} \mathcal{F}_1$ is σ-algebra containing $\mathcal{C}$

Proof: $\mathcal{C} \subseteq \mathcal{P}(\Omega)$
then $\mathcal{C} \subseteq A \subseteq \mathcal{P}(\Omega)$ A is an algebra
from previous result
$\cap A$ is an algebra
so, $\mathcal{A}(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq A \subseteq \mathcal{P}(\Omega)} A$ is an algebra

similarly for $\mathcal{F}_1(\mathcal{C})$ case
as from previous result
every F σ-algebra s.t

$$\mathcal{C} \subseteq F \subseteq \mathcal{P}(\Omega)$$
$$\Rightarrow \mathcal{F}_1(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq F \subseteq \mathcal{P}(\Omega)} F \text{ is } \sigma\text{-algebra}$$

Def: $A(\mathcal{C})$ is the algebra generated by $\mathcal{C}$

$\mathcal{F}(\mathcal{C})$ is the σ-algebra generated by $\mathcal{C}$

Def: Let Ω be a metric space or a topological space. Let $\mathcal{C}$ be collection of all open subsets of Ω. Then

$\mathcal{F}(\mathcal{C})$ is called Borel σ-algebra denoted by $\mathcal{B}(\Omega)$ / $\mathcal{FB}$

Eg: we have to show $\mathcal{F}(S) = \mathcal{B}(\mathbb{R})$

$$S = \{(a,b] \mid -\infty < a,b < \infty\}$$

for $(a,b] \in S$

$\Rightarrow (a,b] = (a,b) \cup \{b\}$

as (a,b) is open set

$\Rightarrow (a,b) \in \mathcal{C}$

$\Rightarrow (a,b) \in \mathcal{B}(\mathbb{R})$

also as $(-\infty, b) \cup (b, \infty) \in \mathcal{B}(\mathbb{R})$

$\Rightarrow \{b\} \in \mathcal{B}(\mathbb{R})$

$\Rightarrow (a,b] \in \mathcal{B}(\mathbb{R})$

$\Rightarrow \mathcal{F}(S) \subseteq \mathcal{B}(\mathbb{R})$

as, $(a,b) \in \mathcal{C}$, $(a,b) = \bigcup_{n \geq n_0} (a, b - \frac{1}{n}] \in \mathcal{F}(S)$ $\left(\because (a, b-\frac{1}{n}] \in S \ \forall n \geq n_0\right)$

$\Rightarrow (a,b) \in \mathcal{F}(S)$

$\Rightarrow \mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}(S)$

so, $\mathcal{B}(\mathbb{R}) = \mathcal{F}(S)$

Theorem: Let $S \subseteq \mathcal{P}(\Omega)$ be semi-algebra, then $A \in \mathcal{A}(S)$ iff

$\exists$ disjoint collection of sets $E_1 \ldots E_n$ in S s.t $A = \bigcup_{i=1}^{n} E_i$

proof: ($\Leftarrow$) is trivial (as $E_i \in S \subseteq \mathcal{A}(S)$)

($\Rightarrow$) consider $\mathcal{G} = \left\{ \bigcup_{j=1}^{m} F_j \;\middle|\; F_j \in S, 1 \leq j \leq m \text{ (disjoint)}, F_j \cap F_k = \phi, j \neq k \right\}$ every element of S

$S \subseteq \mathcal{G} \subseteq \mathcal{A}(S)$ (trivial, trivial)

if we show $\mathcal{G}$ is algebra, then we get $\mathcal{G} = \mathcal{A}(S)$

as, $\Omega \in \mathcal{G}$, let $A, B \in \mathcal{G}$ then $A = \bigcup_{i=1}^{m} E_i$ $\{E_i\}_{i=1}^{m} \subseteq S$ disjoint

$B = \bigcup_{j=1}^{m} F_j^\circ \quad \{F_j^\circ\}_{j=1}^{m} \subseteq S$ disjoint

then $A \cap B = \bigcup_{i=1}^{m}\bigcup_{j=1}^{m}(E_i \cap E_j)$ and

$\{E_i \cap F_j\}_{(i,j)=(1,1)}^{(m,n)}$ is disjoint collection

of elements of S

$\Rightarrow A \cap B \in \mathcal{G}_M$

$\cap \left\{ \begin{matrix} \cup F_{1,1}\ F_{1,2} \ldots F_{1,n_1} \\ \cup F_{2,1} \ldots\ldots F_{2,n_2} \\ \vdots \\ \cup F_{m,1} \ldots F_{m,n_m} \end{matrix} \right\}$

$\forall\ 1 \le p \le M,\quad E_p^c = \bigcup_{K=1} F_{p,K}^\circ$ ($\because S$ is semi algebra)

intersection of disjoint ~ all small elements union

$\{F_{i,K}\}_{K=1}^{m}$ is disjoint in S, $A^c = \bigcap_{i=1}^{m}\left(\bigcup_{K=1}^{n_p} F_{i,K}\right) = \bigcup_{K_1=1}^{n_1} \cdots \bigcup_{K_m=1}^{n_m} (F_{1,K_1} \cap \cdots F_{m,K_m}) \in \mathcal{G}$, so

$\mathcal{G} = \mathcal{A}(S)$

Exe: An algebra $\mathcal{A}$ is also closed under finite unions

Ans:

$\mathcal{A}$ is an algebra, then

as $\forall A, B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$ ($\because$ Property)

and $A^c, B^c \in \mathcal{A}$ ($\because$ property)

then, as $A^c \in \mathcal{A}$, $B^c \in \mathcal{A}$

$\Rightarrow A^c \cap B^c \in \mathcal{A}$

$\Rightarrow (A^c \cap B^c)^c \in \mathcal{A}$ ($\because$ demorgan law)

$\Rightarrow A \cup B \in \mathcal{A}$

so, $\forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

then now for $n=2$: $\{A_i\}_{i=1}^{n}$ s.t $A_i \in \mathcal{A}\ \forall i$

$\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ for $n=2$

Let's assume true for $n=K+1$,

$n=K$: $\{A_i\}_{i=1}^{K}$ we have $\bigcup_{i=1}^{K-1} A_i \in \mathcal{A}$ and $A_K \in \mathcal{A}$

then from $n=2$ case

$\left(\bigcup_{i=1}^{K-1} A_i\right) \cup (A_K) \in \mathcal{A}$

$\Rightarrow \bigcup_{i=1}^{K} A_i \in \mathcal{A}$

so, by induction, true for all n

Exe: A σ-algebra is closed under countable union

Ans:

now, if $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}_1$ ($\mathcal{F}_1$ is σ-algebra)

then from property of σ-algebra

$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}_1$

also $\forall A \in \mathcal{F}_1 \Rightarrow A^c \in \mathcal{F}_1$ ($\because$ property)

so, $\{A_i^c\}_{i=1}^{\infty} \subseteq \mathcal{F}_1 \Rightarrow \bigcap_{i=1}^{\infty} A_i^c \in \mathcal{F}_1$

$$\Rightarrow \left(\bigcap_{i=1}^{\infty} A_i^{c}\right)^{c} \in \mathcal{F}_1$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1 \quad (\because \text{demorgan law})$$

Exe: 1) Let $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ be a collection of algebra on Ω
($\mathcal{A}_\lambda \subseteq P(\Omega)$ is an algebra $\forall \lambda \in \Lambda$)

show that $\bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda \subseteq P(\Omega)$ is an algebra

2) Let $\{\mathcal{F}_\lambda \mid \lambda \in \Lambda\}$ be a collection of σ-algebra in Ω, show that $\bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda \subseteq P(\Omega)$ is σ-algebra

3) give counter example in case of semi algebra

Ans:

1) $\{\mathcal{A}_\lambda \mid \lambda \in \Lambda\}$ is a collection of algebra on Ω s.t. $\mathcal{A}_\lambda \subseteq P(\Omega)$ $\forall \lambda \in \Lambda$ is an algebra

then as $\mathcal{A}_\lambda \subseteq P(\Omega)$ $\forall \lambda \in \Lambda$

$$\Rightarrow \bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda \subseteq P(\Omega) \quad \text{---(1)}$$

also, as $\forall \lambda \in \Lambda$, $\Omega \in \mathcal{A}_\lambda$ ($\because$ property of Algebra)

$$\Rightarrow \Omega \in \bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda \quad \text{---(2)}$$

then, if $A, B \in \bigcap_{\lambda \in \Omega} \mathcal{A}_\lambda$ then $A, B \in \mathcal{A}_\lambda$ $\forall \lambda \in \Lambda$

$\Rightarrow A \cap B \in \mathcal{A}_\lambda$ $\forall \lambda \in \Omega$ ($\because$ property)

$\Rightarrow A \cap B \in \bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda$ ---(3)

if, $A \in \bigcap_{\lambda \in \Omega} A_\lambda$ then $A \in \mathcal{A}_\lambda$ $\forall \lambda \in \Omega$

$\Rightarrow A^c \in \mathcal{A}_\lambda$ $\forall \lambda \in \Omega$ ($\because$ property)

$\Rightarrow A^c \in \bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda$ ---(4)

from (1), (2), (3), (4) $\bigcap_{\lambda \in \Lambda} A_\lambda$ is an algebra

2) $\{\mathcal{F}_\lambda \mid \lambda \in \Lambda\}$

now similar to above as $\mathcal{F}_\lambda \subseteq P(\Omega)$ $\forall \lambda \in \Omega$

$$\Rightarrow \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda \subseteq P(\Omega) \quad \text{---(1)}$$

also, $\Omega \in \mathcal{F}_\lambda$ $\forall \lambda \in \Lambda$ ($\because$ property)

$$\Rightarrow \Omega \in \bigcap_{\lambda \in \Omega} \mathcal{F}_\lambda \quad \text{---(2)}$$

now, for $\{A_i\}_{i=1}^{\infty} \in \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$

$\Rightarrow \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}_\lambda$ $\forall \lambda \in \Lambda$

$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}_\lambda$ $\forall \lambda \in \Lambda$ ($\because$ property)

$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ ---(3)

also, $\forall A \in \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda \Rightarrow A \in \mathcal{F}_\lambda \ \forall \lambda \in \Lambda$
$\Rightarrow A^c \in \mathcal{F}_\lambda \ \forall \lambda \in \Lambda$ (∵ property)
$\Rightarrow A^c \in \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ — (4)

from (1), (2), (3), (4) $\bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda \subseteq P(\Omega)$ is a σ-algebra

3) let $\Omega = \{1,2,3,4,5,6\}$

and so, let $A_1 = \{\phi, \Omega, \{1,2\}, \{3,4\}, \{5,6\}\}$
then
trivial to see all properties are satisfied

$A_2 = \{\phi, \Omega, \{1,2\}, \{3,5\}, \{4,6\}\}$
trivial to see these also

but $A_1 \cap A_2 = \{\phi, \Omega, \{1,2\}\}$, $\{1,2\} \in A_1 \cap A_2$
but
$\{1,2\}^c = \{3,4,5,6\}$ not a union
of any elements of $A_1 \cap A_2$

so, $A_1 \cap A_2$ not a semi algebra

12th Jan:

Defn: Let $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ with $\phi \in \mathcal{C}$, A function $\mu: \mathcal{C} \to [0,\infty]$ is called an additive set function if

① $\mu(\phi) = 0$

② if $\{E_i\}_{i=1}^{n} \subseteq \mathcal{C}$ is a disjoint collection s.t $\bigcup_{i=1}^{n} E_i \in \mathcal{C}$ then

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$$

Defn: Let $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ with $\phi \in \mathcal{C}$, a function $\mu: \mathcal{C} \to [0,\infty]$ is called σ-additive set function or measure if

① $\mu(\phi) = 0$

② if $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}$ is a disjoint collection s.t $\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Note: in finite sets, $\mu(\phi)=0$ will be implied as $\mu(A) = \mu(A) + \mu(\phi)$

also $\mu(F) = \mu(F \setminus E \cup E) = \mu(F \setminus E) + \mu(E)$ $E \subseteq F$

if $\mu(F) = \infty$ then $\mu(E) \leq \infty$ (trivial), if $\mu(F) < \infty$ then $\mu(E) \leq \mu(F)$

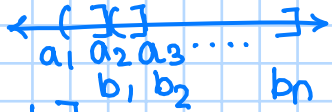
Eg: Let $\Omega = (0,1]$, this collection $\mathcal{C} = \{(a,b] \text{ or } a < b \leq 1\} \cup \{\phi\} \subseteq \mathcal{P}(\Omega)$

Let $\mu: \mathcal{C} \to [0,\infty]$ by $\mu((a,b]) = \infty$ if $a = 0$

$= b - a$ if $a > 0$

now let $\{(a_i,b_i]\}_{i=1}^{n} \subseteq \mathcal{C}$ be a disjoint collection s.t

$\bigcup_{i=1}^{n} (a_i,b_i] \in \mathcal{C}$



then $\bigcup_{i=1}^{n} (a_i,b_i] = (a,b]$

Case I: $a_1 = 0$ then $\mu\left(\bigcup_{i=1}^{n} (a_i,b_i]\right) = \infty = \sum_{i=1}^{n} \mu((a_i,b_i])$ ← as $\mu((a_1,b_1]) = \infty$

Case II: $a_1 \neq 0$ then $\mu((a,b]) = b - a$

$= b - a_n + a_n - a$

$= b - a_n + b_{n-1} - a$

$\vdots$

$= \sum_{j=1}^{n} \mu((a_j,b_j])$

and now $\mu((\frac{1}{2},\frac{3}{4}]) = \mu((\frac{1}{2},\frac{3}{4}]) + \mu(\phi)$

$\Rightarrow 0 = \mu(\phi)$

so, μ is additive function

μ is not σ-additive as

$I = (0,\frac{1}{2}]$ $\quad I_j = \left(\frac{1}{2^{(j+1)}}, \frac{1}{2^j}\right]$ $\forall j \in \mathbb{N}$

$\bigcup_{j=1}^{\infty} I_j = I = (0,\frac{1}{2}]$

$\mu(I) = \infty$ ($\because$ by definition)

$$\mu(I_j^\circ) = \frac{1}{2^j} - \frac{1}{2(j+1)}$$

$$\sum_{j=1}^{\infty} \mu(I_j^\circ) = \frac{1}{2} - \frac{1}{2(2)} + \frac{1}{2(2)} - \frac{1}{2(3)} + \cdots = \frac{1}{2} \neq \infty$$

Exe: let $\Omega = \mathbb{R}$ and let $S = \{(a,b] \mid -\infty < a < b < \infty\}$ and define $\lambda : S \to [0,\infty]$ by — S is a semi algebra → love down

$$\lambda((a,b]) = \begin{cases} b-a \text{ ; if } a,b \in \mathbb{R} \\ \infty \text{ : otherwise} \end{cases}$$

show that λ is a σ-additive set function/measure

Notations:

let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\Omega)$ and $E \subseteq \Omega$ then

(i) $E_n \uparrow E$ if $E_n \subseteq E_{n+1}$ $\forall n \in \mathbb{N}$ and $E = \bigcup_{n=1}^{\infty} E_n$

(ii) $E_n \downarrow E$ if $E_{n+1} \subseteq E_n$ $\forall n \in \mathbb{N}$ and $E = \bigcap_{n=1}^{\infty} E_n$

Def: let $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ and let $\mu : \mathcal{C} \to [0,\infty]$ be any function then

1) if $E \in \mathcal{C}$ then μ is said to be continuous from below at E if $\forall \{E_n\} \subseteq \mathcal{C}$ with $E_n \uparrow E$

$$\mu(E_n) \to \mu(E) \text{ as } n \to \infty$$

2) if $E \in \mathcal{C}$ then μ is said to be continuous from above at E if $\forall \{E_n\} \subseteq \mathcal{C}$ with $E_n \downarrow E$ and $\exists n_0$ s.t $\mu(E_{n_0}) < \infty$

$$\mu(E_n) \to \mu(E) \text{ as } n \to \infty$$

Note: above in 2) we have extra conditions as say

$$\Omega = \mathbb{R}, \quad S = \{(a,b] \mid -\infty < a < b < \infty\}$$
$$\lambda : S \to [0,\infty]$$

let $E_n \downarrow \phi$ but $\mu(E_n) \not\to 0$ as $n \to \infty$

↖ if $\mu(E_n) = \infty$ (one extreme case)

let $E_n = (-\infty, -n]$ $E_n \downarrow \phi$

$\mu(E_n) = \infty \not\to 0$ as $n \to \infty$

$\bigcap_{n=1}^{\infty} E_n = \phi$ as open on left $E = \phi \in S$

Def: If $E \in \mathcal{C}$, then μ is said to be continious at E if μ is continious from above and below at E

Lemma: let $\mathcal{A} \subseteq P(\Omega)$ be an algebra and let $\mu: \mathcal{A} \to [0,\infty]$ be an additive set fution, then

1) if μ is σ-additive, then μ is continious at all $E \in \mathcal{A}$

2) if μ is continious fution $\forall E \in \mathcal{A}$ then μ is σ-additive

3) if μ is continious from above at ϕ with $\mu(\Omega) < \infty$, then μ is σ-additive

proof: 1) let $E \in \mathcal{A}$ and $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ s.t $E_n \uparrow E$, we use disjointification:

$F_1 = E_1 \quad \dots \quad F_K = E_K \setminus E_{K-1} \quad \forall K \geq 2$

then $\{F_K\}_{K=1}^{\infty} \subseteq \mathcal{A}$ is disjoint collection and

$E_n = \bigcup_{K=1}^{n} F_K$

so, $\bigcup_{K=1}^{\infty} F_K = E$

by σ-additivity $\mu(E) = \mu\left(\bigcup_{K=1}^{\infty} F_K\right) = \sum_{K=1}^{\infty} \mu(F_K)$

$= \lim_{n\to\infty} \sum_{K=1}^{n} \mu(F_K)$

$= \lim_{n\to\infty} \mu\left(\bigcup_{K=1}^{n} F_K\right)$

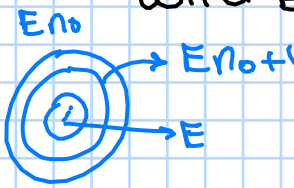
so, $\mu(E) = \lim_{n\to\infty} \mu(E_n)$

so, μ is continious from below at E, now

if $E \in \mathcal{A}$, $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ s.t $E_n \downarrow E$ and $\exists n_0$ s.t $\mu(E_{n_0}) < \infty$

let $G_K = E_{n_0} \setminus E_{n_0+K}$ then $\{G_K\}_{K=1}^{\infty} \subseteq \mathcal{A}$ and $G_K \uparrow (E_{n_0} \setminus E)$

with $E_{n_0} \setminus E \in \mathcal{A}$



as $E_{n_0+K+1} \subseteq E_{n_0+K}$

$\Rightarrow E_{n_0} \setminus E_{n_0+K} \subseteq E_{n_0} \setminus E_{n_0+K+1}$

$\Rightarrow G_K \subseteq G_{K+1}$

and $\bigcup_{K=1}^{\infty} G_K = E_{n_0} \setminus E$

so, $\lim_{n\to\infty} \mu(G_K) = \mu(E_{n_0} \setminus E)$ (∵ above calculation)

then $\mu(G_K) = \mu(E_{n_0} \setminus E_{n_0+K})$

$\mu(E_{n_0}) = \mu(E_{n_0} \setminus E_{n_0+K} \cup E_{n_0+K})$

$= \mu(E_{n_0} \setminus E_{n_0+K}) + \mu(E_{n_0+K})$ (∵ both disjoint and in $\mathcal{A}$)

$\Rightarrow \mu(E_{n_0}) - \mu(E_{n_0+K}) = \mu(E_{n_0} \setminus E_{n_0+K})$

similar for $\mu(E_{n_0}) - \mu(E) = \mu(E_{n_0} \setminus E)$

so, $\lim_{n\to\infty} \mu(G_n) = \lim_{n\to\infty} \mu(E_{n_0}) - \mu(E_{n_0+n}) = \mu(E_{n_0}) - \lim_{n\to\infty} \mu(E_{n_0+n})$

$= \mu(E_{n_0}) - \mu(E)$

$\Rightarrow \lim_{n\to\infty} \mu(E_n) = \mu(E)$

so, σ-additive $\Rightarrow \mu$ is continuous $\forall E \in \mathcal{A}$

Eg: let $\Omega = \mathbb{R}$ and let $S = \{(a,b] \mid -\infty \leq a \leq b \leq \infty\}$ and define $\lambda: S \to [0,\infty]$ by

S is a semi algebra

$$\lambda((a,b]) = \begin{cases} b-a & \text{; if } a,b \in \mathbb{R} \\ \infty & \text{: otherwise} \end{cases}$$

show that λ is a σ-additive set function/measure

Ans:

let $\{E_i\}_{i=1}^{\infty} \subseteq S$ be pairwise disjoint

s.t $E = \bigcup_{i=1}^{\infty} E_i \in S$

then E is of form

$E = (a,b] \quad -\infty \leq a, b \leq \infty$

as $E_i \in S \quad E_i = (a_i, b_i]$ s.t $-\infty \leq a_i, b_i \leq \infty \ \forall i \in \mathbb{N}$

then $a = \inf\{a_i\}_{i=1}^{\infty} \quad b = \sup\{b_i\}_{i=1}^{\infty}$ as

E_i's are disjoint so E would be s.t $(a,b]$ where a is lowest of all $\{a_i\}_{i=1}^{\infty}$, and b to be largest of all $\{b_i\}_{i=1}^{\infty}$. sup and inf will always exist as extended $\mathbb{R}$

now, $\forall i \in \mathbb{N}, \ E_i = (a_i, b_i] \ \exists j \in \mathbb{N}$ s.t

$E_j = (b_i, c_i]$ or $E_j = (c_i, a_i]$

as if not then $\bigcup_{i=1}^{\infty} E_i = (\ ] \cup (\ ]$ ← there will be a gap in between

so now if we pick E_1 from $\{E_i\}_{i=1}^{\infty}$

then $\exists i_1$ s.t $E_{i_1} = (b_1, c_1]$ or $E_{i_1} = (c_1, a_1]$

so, $E_1 \cup E_{i_1} = F_1 \in S$ by definition

then now remove E_1, E_{i_1} from $\{E_i\}_{i=1}^{\infty}$ and put F_1

so let this set be $\mathcal{C}_1 = \{F_1, E_2, E_3 \ldots\} \setminus \{E_1\}$

now do the same for F_1 i.e

$\exists i_2$ s.t $F_1 \cup E_{i_2} \in S$

let $F_1 \cup E_{i_2} = F_2$

consist $\mathcal{C}_2 = \{F_2, E_3, \ldots\} \setminus \{E_{i_1}, E_{i_2}\}$

repeate this process to get

$$\mathcal{C}_n = \{F_n, E_{n+1}, \dots\} \setminus \{E_{i_1}, E_{i_2}, \dots E_{i_n}\}$$

then as $F_n = \bigcup_{r=1}^{n} (E_{i_r}) \cup E_1$ by construction

so, $\lim_{n\to\infty} F_n = \bigcup_{i=1}^{\infty} E_i$

also, $\lambda(E_1) = b_1 - a_1$ (or ∞ if b_1 or $a_1 \notin \mathbb{R}$)

then as $F_1 = E_1 \cup E_{i_1}$

wlog $b_1 = a_{i_1}$ then

$F_1 = (a_1, b_{i_1}]$

$\lambda(F_1) = b_{i_1} - a_1$ (or ∞ if b_{i_1} or $a_1 \notin \mathbb{R}$)

$= (b_{i_1} - a_{i_1}) + (b_1 - a_1)$

$= \lambda(E_1) + \lambda(E_{i_1})$ (or ∞ trivial case)

$\vdots$ (repeating steps)

$$\lambda(F_n) = \lambda(E_1) + \sum_{r=1}^{n} \lambda(E_{i_r})$$

$$\lim_{n\to\infty} \lambda(F_n) = \lim_{n\to\infty} \left(\lambda(E_1) + \sum_{r=1}^{n} \lambda(E_{i_r}) \right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{N} \lambda(E_i)$$

$$\Rightarrow \lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i)$$

so, $\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i)$

as $(a,b]$ for $a = b = 0 \in \mathbb{R}$

we have

$(0,0] = \phi$

$\Rightarrow \lambda(\phi) = b - a = 0$

so, λ is σ-additive

Exe: try to see why extra hypothesis is required in (3) for above lemma

Ans:

if $\mu(\Omega) = \infty$ and we can force $\mathcal{A} \subseteq \mathcal{C}\{(a,b] \mid 0 \le a < b < 1\}$ to be s.t. $\lambda((a,b]) = \begin{cases} b-a; & a \neq 0 \\ \infty; & a = 0 \end{cases}$

$\Omega = (0,1)$, $\mathcal{C} \subseteq \mathcal{P}(\Omega)$, $\cup\phi$ with other properties that make it additive

then $\forall \{E_i\}_{i=1}^{\infty}$ s.t. $E_i \downarrow E = \phi$ we will have $\mu(E_i) \to \mu(\phi) = 0$ as E_i will never be of form $(0,0]$ as $a < b$

but μ is not σ-additive is already seen before

15th Jan:

Lemma: let $\mathcal{A} \subseteq P(\Omega)$ be an algebra and let $\mu : \mathcal{A} \to [0,\infty]$ be an additive set function, then

1) if μ is σ-additive, then μ is continuous at all $E \in \mathcal{A}$

2) if μ is continuous function from below, $\forall E \in \mathcal{A}$ then μ is σ-additive

3) if μ is continuous from above at ϕ with $\mu(\Omega) < \infty$, then μ is σ-additive

proof: (1) μ is σ-additive $\Rightarrow$ μ is cont from above $\forall E \in \mathcal{A}$

now for $E \in \mathcal{A}$

$$\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A},\ E_{n+1} \subseteq E_n\ \forall n,\ \exists n_0 \text{ s.t } \mu(E_{n_0}) < \infty$$

$$E = \bigcap_{n=1}^{\infty} E_n$$

let $G_K = E_{n_0} \setminus E_{n_0+K}$ $\forall K \in \mathbb{N}$, as $\mathcal{A}$ is algebra

$$\{G_K\}_{K=1}^{\infty} \subseteq \mathcal{A} \text{ and}$$

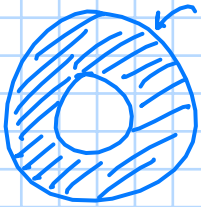
$G_K \uparrow E_{n_0} \setminus E \in \mathcal{A}$ as $K \to \infty$ ($E_{n_0} \setminus E \in \mathcal{A}$ is trivial)

then using first part we get

$$\lim_{K\to\infty} \mu(G_K) = \mu(E_{n_0} \setminus E)$$

$$= \lim_{K\to\infty} \mu(E_{n_0} \setminus E_{n_0+K})$$

as $\mu(E_{n_0}) < \infty$ and $E_{n_0+K} \subseteq E_{n_0}$



$$\text{so, } \mu(E_{n_0} \setminus E) = \lim_{K\to\infty} (\mu(E_{n_0}) - \mu(E_{n_0+K}))$$

$$\mu(E_{n_0}) - \mu(E) = \mu(E_{n_0}) - \lim_{K\to\infty} \mu(E_{n_0+K})$$

$$\Rightarrow \mu(E) = \lim_{K\to\infty} \mu(E_{n_0+K}) = \lim_{K\to\infty} \mu(E_K)$$

$$\Rightarrow \forall E \text{ s.t } E_n \downarrow E,\ \lim_{n\to\infty} \mu(E_n) = \mu(E)$$

so, μ is continuous from above, $\forall E \in \mathcal{A}$

so, if μ is σ-additive it is continuous $\forall E \in \mathcal{A}$

(2) let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a disjoint collection

let $E = \bigcup_{n=1}^{\infty} E_n$ and define $F_n = \bigcup_{K=1}^{n} E_K$ and then

$F_n \uparrow E$, now as μ is continuous from below

$$\mu(E) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu\left(\bigcup_{K=1}^{n} E_K\right) = \lim_{n\to\infty} \sum_{K=1}^{n} \mu(E_K)$$

$$\Rightarrow \mu(E) = \sum_{n=1}^{\infty} \mu(E_n) \qquad (\because \mu \text{ is additive})$$

(3) μ is continious from above at ϕ and $\mu(\Omega)<\infty$

$\{E_n\}_{n=1}^{\infty}$ is disjoint collection, let $E=\bigcup_{n=1}^{\infty}E_n$

define $G_n=\bigcup_{k>n-1}^{\infty}E_k=E\setminus\left(\bigcup_{k=1}^{n+1}E_k\right)$ then $G_n\downarrow\phi$ by hypothesis

as $\mu(\Omega)<\infty\Rightarrow\mu(G_1)<\infty$, so $\exists n_0=1$ s.t $\mu(G_{n_0})<\infty$

$\mu(G_n)\longrightarrow 0$ as $n\longrightarrow\infty$

$$\Rightarrow \lim_{n\to\infty}\mu(G_n)=0$$

now, $\mu(E)=\mu\left(\bigcup_{n=1}^{\infty}E_n\right)$

$$=\mu\left(\bigcup_{k=1}^{n-1}E_k\right)+\mu(G_k)$$

$$=\sum_{k=1}^{n+1}\mu(E_k)+\mu(G_k)$$

$$\Rightarrow \mu(E)=\lim_{n\to\infty}\sum_{k=1}^{\infty}\mu(E_k)+\lim_{n\to\infty}\mu(G_k)\quad\left(\text{as }\lim_{n\to\infty}\sum_{k=1}^{\infty}\mu(E_k)<\infty\right)$$

as $\mu(\Omega)<\infty$

Exe: if $\lim_{n\to\infty}\mu(G_k)=0$ and $\mu(E)=\infty$, show that

if $\mu(E)=\sum_{n=1}^{k}\mu(E_k)+\mu(G_k)$ then

$\mu(E)=\lim_{n\to\infty}\sum_{n=1}^{k}\mu(E_k)=\sum_{n=1}^{\infty}\mu(E_k)$ → done down

Theorm: Let $S\subseteq P(\Omega)$ be semi-algebra and let $\mu:S\longrightarrow[0,\infty]$ be additive (resp σ-additive) set fution, then $\exists$ a set fution

$\nu:\mathcal{A}(S)\longrightarrow[0,\infty]$ which is additie (resp. σ-additie)

s.t

1) $\nu(A)=\mu(A)\quad\forall A\in S$

2) If $\exists$ two additie fuctions (resp. σ-additie) set fuctions $\nu_1,\nu_2:\mathcal{A}(S)\longrightarrow[0,\infty]$ satisfying (1) then $\nu_1(E)=\nu_2(E)\ \forall E\in\mathcal{A}(S)$

proof: Suppose $A\in\mathcal{A}(S)$ then $\exists\{E_k\}_{k=1}^{n}\subseteq S$ disjoint collection

s.t $A=\bigcup_{k=1}^{n}E_k$ (∵ we have shown this before)

then let $\nu:\mathcal{A}(S)\longrightarrow[0,\infty]$ by

$$\nu(A)=\sum_{k=1}^{n}\mu(E_k)$$

as $A=\bigcup_{k=1}^{n}E_k$ decomposition is not unique, we have to show $\nu(A)$ is well defined

if decomposition of $A=\bigcup_{l=1}^{m}F_l$ $\{F_l\}_{l=1}^{m}\subseteq S$ is disjoint

observe that $E_k=E_k\cap A$ and now use

$$=E_k\cap\left(\bigcup_{l=1}^{m}F_l\right)$$

$$= \bigcup_{l=1}^{m} (E_k \cap F_l)$$

$$\Rightarrow A = \bigcup_{k=1}^{n} E_k$$

$$= \bigcup_{k=1}^{n} \bigcup_{l=1}^{m} E_k \cap F_l$$

↝ Each of this set is disjoint as 2 different pairs

we will prove this later

Suppose μ is additive $\mu(E_k) = \mu(\bigcup_{l=1}^{m} E_k \cap F_l)$

$$= \sum_{l=1}^{m} \mu(E_k \cap F_l)$$

$$\Rightarrow \nu(A) = \sum_{k=1}^{n} \mu(E_k) = \sum_{k=1}^{n} \sum_{l=1}^{m} \mu(E_k \cap F_l)$$

$$= \sum_{l=1}^{m} \sum_{k=1}^{n} \mu(E_k \cap F_l)$$

$$= \sum_{l=1}^{m} \mu(\bigcup_{k=1}^{n} E_k \cap F_l)$$

$$= \sum_{l=1}^{m} \mu(F_l)$$

$$\Rightarrow \sum_{k=1}^{n} \mu(E_k) = \sum_{l=1}^{m} \mu(F_l)$$

So, ν is well-defined

now to show ν is additive, let $A, B \in \mathcal{A}(S)$ s.t $A \cap B = \emptyset$

consider $A = \bigcup_{k=1}^{n} E_k$, $B = \bigcup_{l=1}^{m} F_l$ where

$\{E_k\}_{k=1}^{n}, \{F_l\}_{l=1}^{m} \subseteq S$ disjoint

then $A \cup B = (\bigcup_{k=1}^{n} E_k) \cup (\bigcup_{l=1}^{m} F_l)$ as $A \cap B = \emptyset$

$\{E_i, F_j\}$ disjoint and $\subseteq S$

by defn $\mu(A \cup B) = \sum_{i=1}^{n} \mu(E_i) + \sum_{j=1}^{m} \mu(F_j)$ as $A \cup B \in \mathcal{A}$ and $A \cup B = (\bigcup_{i=1}^{n} E_i) \cup (\bigcup_{j=1}^{m} F_j)$

(2) Suppose $\nu_1, \nu_2 : \mathcal{A}(S) \longrightarrow [0, \infty]$ are additive set functions s.t

$\nu_1(E) = \mu(E) = \nu_2(E)$ $\forall E \in S$

Let $A \in \mathcal{A}(S)$ with $A = \bigcup_{i=1}^{n} E_k$ where $\{E_k\}_{k=1}^{n} \subseteq S$ is finite disjoint collection

then, $\nu_1(A) = \nu_1(\bigcup_{i=1}^{n} E_k) = \sum_{k=1}^{n} \nu_1(E_k)$ as $E_k \in S$ $\forall k \in \{1, \dots n\}$

$$= \sum_{k=1}^{n} \mu(E_k)$$

similarly, $\nu_2(A) = \nu_2(\bigcup_{i=1}^{n} E_k) = \sum_{k=1}^{n} \nu_2(E_k) = \sum_{k=1}^{n} \mu(E_k)$

$\Rightarrow \nu_1(A) = \nu_2(A)$ $\forall A \in \mathcal{A}(S)$ as A is chosen arbitrarily

now if μ is σ-additive then, to show that ν as defined before is also σ-additive

suppose $A \in \mathcal{A}(S)$ and $A = \bigcup_{j=1}^{\infty} A_j$ (well defined argument is same)

$\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{A}(S)$ disjoint.

Then we have to show $\mu(A) = \sum_{j=1}^{\infty} \nu(A_j)$

now let $A = \bigcup_{j=1}^{m} E_j^\circ$ where $\{E_j\}_{j=1}^{m} \subseteq S$ disjoint and

$A_K = \bigcup_{l=1}^{n_K} E_{K,l}$ $\{E_{K,l}\}_{l=1}^{n_K} \subseteq S$ disjoint $\forall K \in \mathbb{N}$

now, $\nu(A) = \sum_{j=1}^{m} \mu(E_j^\circ)$

$E_j^\circ = E_j^\circ \cap A$

$= E_j^\circ \cap \left(\bigcup_{j=1}^{\infty} A_j\right)$

$= E_j \cap \left(\bigcup_{K=1}^{\infty} \bigcup_{l=1}^{n_K} E_{K,l}\right)$

$\Rightarrow E_j^\circ = \bigcup_{K=1}^{\infty} \bigcup_{l=1}^{n_K} (E_j \cap E_{K,l})$ ← disjoint

$\Rightarrow \mu(E_j) = \mu\left(\bigcup_{K=1}^{\infty} \bigcup_{l=1}^{n_K} (E_j \cap E_{K,l})\right)$

$= \sum_{K=1}^{\infty} \sum_{l=1}^{n_K} \mu(E_j \cap E_{K,l})$

$\Rightarrow \nu(A) = \sum_{j=1}^{m} \mu(E_j) = \sum_{j=1}^{m} \sum_{K=1}^{\infty} \sum_{l=1}^{n_K} \mu(E_j \cap E_{K,l})$ —— ①

On the other hand, $\nu(A_K) = \sum_{l=1}^{n_K} \mu(E_{K,l})$

as before $\mu(E_{K,l}) = \mu(E_{K,l} \cap A)$

$= \sum_{j=1}^{m} \mu(E_{K,l} \cap E_j)$

$\Rightarrow \sum_{l=1}^{n_K} \mu(E_{K,l}) = \sum_{l=1}^{n_K} \sum_{j=1}^{m} \mu(E_{K,l} \cap E_j) = \nu(A_K)$

then $\sum_{K=1}^{\infty} \nu(A_K) = \sum_{K=1}^{\infty} \sum_{l=1}^{n_K} \sum_{j=1}^{m} \mu(E_{K,l} \cap E_j) = \sum_{K=1}^{\infty} \sum_{j=1}^{m} \sum_{l=1}^{n_K} \mu(E_j \cap E_{K,l})$

($\because$ disjoint, so finite sum interchanged)

now, $\sum_{K=1}^{\infty} \sum_{j=1}^{m} \left(\sum_{l=1}^{n_K} \mu(E_j \cap E_{K,l}) \right) = \alpha$

Case I: $\alpha \in \mathbb{R}$ then series converges and so we can interchange limit to get

$\alpha = \sum_{j=1}^{m} \sum_{K=1}^{\infty} \sum_{l=1}^{n_K} \mu(E_j \cap E_{K,l}) = \mu(A)$ ($\because$ ①)

Case II: $\alpha = \infty$: in this case as series is ∞, every term is positive so, if we change limit, if we sum over j, we will still get value to be ∞ as $\exists j \in \{1,2 \dots m\}$ s.t ∞ is result as $j \in \mathbb{N}$, so, we can use order to get $\mu(A)$ from ①

Exe: if $\lim_{n\to\infty} \mu(G_k) = 0$ and $\mu(E) = \infty$, show that if $\mu(E) = \sum_{n=1}^{k} \mu(E_k) + \mu(G_k)$ then

$$\mu(E) = \lim_{n\to\infty} \sum_{n=1}^{k} \mu(E_k) = \sum_{n=1}^{\infty} \mu(E_k)$$

Ans: $\mu(E) = \sum_{n=1}^{k} \mu(E_k) + \mu(G_k)$

now for all $\varepsilon > 0$, if $\exists N \in \mathbb{N}$ s.t $\forall n \geq N \quad x_n > \varepsilon$ then $\lim_{n\to\infty} x_n = \infty$

now, let $x_k = \sum_{n=1}^{k} \mu(E_n)$ then

$$\mu(E) - \mu(G_k) = \sum_{n=1}^{k} \mu(E_n)$$

as $\mu(G_k) \to 0$

$\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t $|\mu(G_k)| < \varepsilon$

$$\Rightarrow -\mu(G_k) > -\varepsilon$$

$$\Rightarrow \sum_{n=1}^{k} \mu(E_n) = \mu(E) - \mu(G_k) > \mu(E) - \varepsilon$$

but as $\mu(E) = \infty > 2\varepsilon$

$$\Rightarrow \sum_{n=1}^{k} \mu(E_n) > \varepsilon$$

So, $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t $x_K > \varepsilon \Rightarrow \lim_{K\to\infty} x_K = \infty$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(E_n) = \mu(E)$$

19th Jan:

Monotone class theorem:

Defn: $\mathcal{M} \subseteq \mathcal{P}(\Omega)$ is called a monotone class if

1) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ with $A_n \uparrow A \Rightarrow A \in \mathcal{M}$

2) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ with $A_n \downarrow A \Rightarrow A \in \mathcal{M}$

Ex: come up with a class that is not a monotone class → done down

Note: ① any σ-algebra is a monotone class

② suppose $\mathcal{M}$ is a monotone class and an algebra then $\mathcal{M}$ is a σ-algebra

let $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ then $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$ (definition of σ-algebra)

let $B_n = \bigcup_{i=1}^{n} A_i$ then $B_n \uparrow A$ and $B_n \in \mathcal{M}\ \forall n \in \mathbb{N}$

as $\mathcal{M}$ is a monotone class and $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ with $B_n \uparrow A \Rightarrow A \in \mathcal{M}$

③ if $\{\mathcal{M}_\alpha \subseteq \mathcal{P}(\Omega) \mid \alpha \in I\}$ of monotone classes, then $\bigcap_{\alpha \in I} \mathcal{M}_\alpha$ is also a monotone class

so, for $\phi \neq \mathcal{C} \subseteq \mathcal{P}(\Omega)$, monotone class generated by $\mathcal{C}$ is defined by

$$\mathcal{M}(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq \mathcal{M} \subseteq \mathcal{P}(\Omega)} \mathcal{M}$$

$\mathcal{M}$ - monotone class

Theorem: (Monotone class theorem) If $\phi \neq \mathcal{A} \subseteq \mathcal{P}(\Omega)$ is an algebra then

$$\mathcal{M}(\mathcal{A}) = \mathcal{F}(\mathcal{A})$$

proof: since $\mathcal{F}(\mathcal{A})$ is a σ-algebra and

$$\mathcal{A} \subseteq \mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{M}(\mathcal{A}) \subseteq \mathcal{F}(\mathcal{A}) \quad \text{——①}$$

($\mathcal{M}(\mathcal{A})$: smallest monotone class containing $\mathcal{A}$; $\mathcal{F}(\mathcal{A})$: σ-algebra is a monotone class)

now, to show $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$

it is sufficient to show that

$\mathcal{M}(\mathcal{A})$ is an algebra

as $\Rightarrow \mathcal{M}(\mathcal{A})$ is σ-algebra ($\because$ monotone and algebra)

$$\Rightarrow \mathcal{F}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$$

($\mathcal{F}(\mathcal{A})$: smallest σ-algebra containing; $\mathcal{M}(\mathcal{A})$: σ-algebra)

now, as $\mathcal{A} \subseteq \mathcal{M}(\mathcal{A})$

$$\Rightarrow \Omega \in \mathcal{M}(\mathcal{A}) \quad \text{——②}$$

let $G = \{A \in \mathcal{M}(\mathcal{A}) \mid \Omega \setminus A \in \mathcal{M}(\mathcal{A})\} \subseteq \mathcal{M}(\mathcal{A})$

then

$\mathcal{A} \subseteq G$ as $\forall A \in \mathcal{A}$, $A^c \in \mathcal{A}$
$\Rightarrow \Omega \setminus A \in \mathcal{M}(\mathcal{A})$
$\Rightarrow A \in G$

so, if we show that G is a monotone class, then
as $G \subseteq \mathcal{M}(\mathcal{A})$
$\Rightarrow G = \mathcal{M}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$
is smallest monotone class)

let $\{B_n\}_{n=1}^{\infty} \subseteq G$ with $B_n \uparrow B$ $B \subseteq \Omega$

by definition, $\{\Omega \setminus B_i\}_{i=1}^{\infty} \subseteq \mathcal{M}(\mathcal{A})$

also, $\Omega \setminus B_n \downarrow \Omega \setminus B$
$\Rightarrow \Omega \setminus B \in \mathcal{M}(\mathcal{A})$
$\Rightarrow B \in G$ ($\because$ definition of G)

similarly if $B_n \downarrow B$ then $\Omega \setminus B_n \uparrow \Omega \setminus B$
$\Rightarrow \Omega \setminus B \in \mathcal{M}(\mathcal{A})$
$\Rightarrow B \in G$

so, G is a monotone class containg $\mathcal{A}$
$\Rightarrow \mathcal{M}(\mathcal{A}) \subseteq G \subseteq \mathcal{M}(\mathcal{A})$
$\Rightarrow \mathcal{M}(\mathcal{A}) = G$

so, $\forall A \in \mathcal{M}(\mathcal{A})$, $A^c \in \mathcal{M}(\mathcal{A})$ ——— (3)

we need to show that $\mathcal{M}(\mathcal{A})$ is closed under finite intersection

first we fix $A \in \mathcal{A}$ and define $H = \{B \in \mathcal{M}(\mathcal{A}) \mid A \cap B \in \mathcal{M}(\mathcal{A})\}$

see that $\forall A' \in \mathcal{A}$, $A' \in \mathcal{M}(\mathcal{A})$
and $A \cap A' \in \mathcal{A} \subseteq \mathcal{M}(\mathcal{A})$
$\Rightarrow A' \in H$
$\Rightarrow \mathcal{A} \subseteq H \subseteq \mathcal{M}(\mathcal{A})$ ← we need $A \in \mathcal{A}$ to get $\mathcal{A} \subseteq H$

now, we will show that H is a monotone class

suppose $\{B_i\}_{i=1}^{\infty} \subseteq H$ and $B \subseteq \Omega$ s.t $B_n \uparrow B$ or $B_n \downarrow B$

as $\mathcal{M}(\mathcal{A})$ is a monotone class
$B \in \mathcal{M}(\mathcal{A})$
by definition of H, $A \cap B_n \in \mathcal{M}(\mathcal{A})$ $\forall n$
then
$A \cap B_n \uparrow A \cap B$ } as $\mathcal{M}(\mathcal{A})$ is a
or $A \cap B_n \downarrow A \cap B$ } monotone class

$\Rightarrow A \cap B \in \mathcal{M}(\mathcal{A})$
$\Rightarrow B \in H$

so, H is a monotone class
$\Rightarrow \mathcal{M}(\mathcal{A}) \subseteq H$
but $H \subseteq \mathcal{M}(\mathcal{A}) \Rightarrow H = \mathcal{M}(\mathcal{A})$

now we fix $A \in \mathcal{M}(\mathcal{A})$ then

$$H' = \{ B \in \mathcal{M}(\mathcal{A}) \mid A \cap B \in \mathcal{M}(\mathcal{A}) \}$$

now also $\forall B \in \mathcal{A}$, $A \cap B \in \mathcal{M}(\mathcal{A})$ is known above

$\Rightarrow B \in H' \quad \forall B \in \mathcal{A}$

$\Rightarrow \mathcal{A} \subseteq H' \subseteq \mathcal{M}(\mathcal{A})$

and similar to above result now

H' is monotone class and so, $H' = \mathcal{M}(\mathcal{A})$

$\Rightarrow \forall A, B \in \mathcal{M}(\mathcal{A})$, $A \cap B \in \mathcal{M}(\mathcal{A})$ ——— (4)

from (2), (3), (4) $\mathcal{M}(\mathcal{A})$ is an algebra

$\Rightarrow \mathcal{M}(\mathcal{A})$ is σ-algebra

then $\mathcal{F}_1(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$ ——— (5) ($\mathcal{M}(\mathcal{A})$: σ-algebra)

($\mathcal{F}_1(\mathcal{A})$: smallest σ-algebra)

so, from (1), (5) $\mathcal{F}_1(\mathcal{A}) = \mathcal{M}(\mathcal{A})$

Caratheodory extension theorm:

The Hypothesis is that

1. $\mathcal{A} \subseteq P(\Omega)$ is an algebra
2. $\nu : \mathcal{A} \longrightarrow [0, \infty]$ is a measure
3. ν is σ-finite

Def: (σ-finite) $\exists \{A_i\}_{i=1}^{\infty} \subseteq P(\Omega)$ s.t $\Omega = \bigcup_{i=1}^{\infty} A_i$ $\nu(A_i) < \infty$, $\forall i \in \mathbb{N}$

Theorm: Under above Hypothesis (1) and (2), $\exists$ a measure $\pi : \mathcal{F}_1(\mathcal{A}) \longrightarrow [0, \infty]$ s.t $\pi|_{\mathcal{A}} = \nu$

furthermore, π is unique if ν is σ-finite (Hypothesis (3))

Def: let $\mathcal{C} \subseteq P(\Omega)$ and $\phi \in \mathcal{C}$, a function

$\mu : \mathcal{C} \longrightarrow [0, \infty]$ is called an outer measure if

1. $\mu(\phi) = 0$
2. countable subadditivity:

if $E \in \mathcal{C}$ and $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{C}$ s.t

$$E \subseteq \bigcup_{n=1}^{\infty} E_n, \text{ then}$$

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Def: let $\nu : \mathcal{A} \longrightarrow [0, \infty]$ be a measure, then define

$$\pi^* : P(\Omega) \longrightarrow [0, \infty] \text{ by } \pi^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \nu(E_i) \;\middle|\; A \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{A}, \forall i \in \mathbb{N} \right\}$$

exe: show that $\pi^* : P(\Omega) \longrightarrow [0, \infty]$ is an outer measure → done down

Exe: come up with a class that is not a monotone class

Ans: class not monotone

is $\{[a,b] \mid -\infty < a,b < \infty\} = C$

as $(0, 1-\frac{1}{n}] = I_n \qquad I_{n+1} = (0, 1-\frac{1}{n+1}]$

then $I_i \in C \ \forall i \in C$

and

$I_i \subseteq I_{i+1}$

so, $I_i \uparrow I = \bigcup_{i=1}^{\infty} I_i = (0,1) \notin C$

so, C is not monotone class

Exe: show that $\pi^*: P(\Omega) \to [0,\infty]$ is an outer measure

Ans:

$$\pi^*: P(\Omega) \to [0,\infty]$$

$$\pi^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \nu(E_i) \;\middle|\; A \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{A},\ \forall i \in \mathbb{N} \right\}$$

① $\pi^*(\phi) = \inf\left\{ \sum_{i=1}^{\infty} \nu(E_i) \;\middle|\; \phi \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{A},\ \forall i \in \mathbb{N} \right\}$

then let $E_i = \phi$ as $E_i \in \mathcal{A}$, $\forall i \in \mathbb{N}$

$$\phi \subseteq \bigcup_{i=1}^{\infty} \phi$$

and $\sum_{i=1}^{\infty} \nu(E_i) = 0$

as $\pi^*: P(\Omega) \to [0,\infty]$

and $\pi^*(\phi) = \inf\{0, \dots\}$

$= 0$

as cannot go below 0 from range of ν and π^*

② now let $\{A_i\}_{i=1}^{\infty} \subseteq P(\Omega)$, $A_i \cap A_j = \phi$ $i \neq j$

$\bigcup_{i=1}^{\infty} A_i \in P(\Omega)$ then

$$\pi^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \inf\left\{ \sum_{i=1}^{\infty} \nu(E_i) \;\middle|\; \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{A},\ \forall i \in \mathbb{N} \right\}$$

$$= \inf\left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \nu(F_{i,j}) \;\middle|\; \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} F_{i,j} \right),\ F_{i,j} \in \mathcal{A},\ \forall i \in \mathbb{N} \right.$$

as always positive and $\left. A_i \subseteq \bigcup_{j=1}^{\infty} F_{i,j} \right\}$

$$\leq \sum_{i=1}^{\infty} \inf\left\{ \sum_{j=1}^{\infty} \nu(F_{i,j}) \;\middle|\; A_i \subseteq \bigcup_{j=1}^{\infty} F_{i,j},\ F_{i,j} \in \mathcal{A},\ \forall i \in \mathbb{N} \right\}$$

($\because$ inf over all sets $\leq$ inf or group addition)

$$\leq \sum_{i=1}^{\infty} \pi^*(A_i)$$

so, π^* is an outer measure

22nd Jan:

$\Omega \neq \phi$, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ - algebra, $\nu : \mathcal{A} \to [0,\infty]$ measure

$$\pi^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \nu(E_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} E_i,\ E_i \in \mathcal{A},\ \forall i \in \mathbb{N} \right\} \quad \forall E \in \Omega$$

Exe: $\pi^* : \mathcal{P}(\Omega) \to [0,\infty]$ is an outer measure

Ans:

observe that $\phi \subseteq \bigcup_{i=1}^{\infty} E_i$ where $E_i = \phi \ \forall i \in \mathbb{N}$

then

$$\pi^*(\phi) \leq \sum_{i=1}^{\infty} \nu(E_i) = 0$$

$$\Rightarrow \pi^*(\phi) \leq 0 \Rightarrow \pi^*(\phi) = 0$$

for subadditivity, let $E \subseteq \bigcup_{i=1}^{\infty} E_i$ $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{P}(\Omega)$

if $\pi^*(E_i) = \infty$ for some i, then we are done

so, we will assume $\forall i \in \mathbb{N}$ $\pi^*(E_i) < \infty$

given $\varepsilon > 0$, $\exists \{H_{i,k}\}_{k=1}^{\infty} \subseteq \mathcal{A}$ s.t. ($\because$ properties of inf)

$$E_i \subseteq \bigcup_{k=1}^{\infty} H_{i,k} \text{ and}$$

$$\pi^*(E_i) \leq \sum_{k=1}^{\infty} \nu(H_{i,k}) \leq \pi^*(E_i) + \frac{\varepsilon}{2^i} \quad \forall i \in \mathbb{N}$$

now, $E \subseteq \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i,k} H_{i,k}$

$$\Rightarrow \pi^*(E) \leq \sum_{i,k} \nu(H_{i,k})$$

$$\leq \sum_{i=1}^{\infty} \left[\pi^*(E_i) + \frac{\varepsilon}{2^i} \right]$$

$$\leq \sum_{i=1}^{\infty} \pi^*(E_i) + \varepsilon \quad \text{(geometric progression)}$$

as $\varepsilon \to 0$, $\pi^*(E) \leq \sum_{i=1}^{\infty} \pi^*(E_i)$

Defn: let $M = \{A \subseteq \Omega \mid \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c) \ \forall E \subseteq \Omega\}$

elements of M are called π^* measurable

Note: ① to show $A \in M$, it is sufficient to verify

$$\pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c) \quad \forall E \in \Omega$$

as $\pi^*(E) \leq \pi^*(E \cap A) + \pi^*(E \cap A^c)$ from defn

② if $A \in M$, then $A^c \in M$

Note: we will see that $\mathcal{P}(\phi) \subset M$ most of time, $M \setminus \mathcal{P}(\phi)$ will have measure 0

Exe: show that $\phi, \Omega \in M$, and $A \in M \Rightarrow A^c \in M$ → done down

Exe: M is an algebra

Ans: we have already seen $\Omega, \phi \in M$ and if $A \in M$, then $A^c \in M$

now, for $A \in M$, $\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c)$

we have to show that for $A, B \in M$, $A \cup B \in M$

i.e $\pi^*(E) \geq \pi^*(E \cap (A \cup B)) + \pi^*(E \cap (A \cup B)^c)$ $\forall E \in \mathcal{P}(\Omega)$

now, let $A \in M$

$$\pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c) \quad \forall E \in \mathcal{P}(\Omega) \text{ —— } (1)$$

$B \in M$, $E \cap A^c \in \mathcal{P}(\Omega)$

$$\pi^*(E \cap A^c) = \pi^*(E \cap A^c \cap B) + \pi^*(E \cap A^c \cap B^c) \text{ —— } (2)$$

from (1), (2)

$$\Rightarrow \pi^*(E) = \pi^*(E \cap A) + \pi^*(E \cap A^c \cap B) + \pi^*(E \cap A^c \cap B^c)$$

now, it is sufficient to show that $\pi^*(E \cap A) + \pi^*(E \cap A^c \cap B) \geq \pi^*(E \cap (A \cup B))$

$A \cup B \subseteq (A) \cup (A^c \cap B)$

$\Rightarrow E \cap (A \cup B) \subseteq (E \cap A) \cup (E \cap (A^c \cap B))$

$\Rightarrow \pi^*(E \cap (A \cup B)) \leq \pi^*(E \cap A) + \pi^*(E \cap (A^c \cap B))$ ($\because$ definition of π^*)

so, $\pi^*(E) \geq \pi^*(E \cap (A \cup B)) + \pi^*(E \cap (A \cup B)^c)$ $\forall E \in \Omega$

$\Rightarrow A \cup B \in M$, so M is an algebra

ext: show $\mathcal{A} \subseteq M$

proof:

let $A \in \mathcal{A}$, to show $\forall E \subseteq \Omega$, $\pi^*(E) \geq \pi^*(E \cap A) + \pi^*(E \cap A^c)$

if $\pi^*(E) = \infty$ then trivial, or else

$\pi^*(E) < \infty$, let $\varepsilon > 0$ then $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t

$$E \subseteq \bigcup_{i=1}^{\infty} E_i \text{ and}$$

$$\pi^*(E) \leq \sum_{i=1}^{\infty} \nu(E_i) \leq \pi^*(E) + \varepsilon$$

then, $\{E_i \cap A\}_{i=1}^{\infty} \subseteq \mathcal{A}$ is a cover for $E \cap A$ and

$\{E_j \cap A^c\}_{j=1}^{\infty} \subseteq \mathcal{A}$ is a cover for $E \cap A^c$

so,

$$\pi^*(E \cap A) + \pi^*(E \cap A^c) \leq \sum_{i=1}^{\infty} \nu(E_i \cap A) + \sum_{i=1}^{\infty} \nu(E_i \cap A^c) \quad (\because \text{definition})$$

$$\leq \sum_{i=1}^{\infty} \left(\nu(E_i \cap A) + \nu(E_i \cap A^c) \right) \quad (\because \text{intersection of } \mathcal{A} \text{ and } \nu \text{ is measure})$$

$$\leq \sum_{i=1}^{\infty} \nu(E_i) \leq \pi^*(E) + \varepsilon$$

let $\varepsilon \to 0$ to get $\pi^*(E\cap A)+\pi^*(E\cap A^c)\le \pi^*(E)$

$\Rightarrow A\in M$

so, $\forall A\in\mathcal{A} \Rightarrow A\in M$

Exe: If $\{F_j\}_{j=1}^n \subseteq M$, is a disjoint collection then

$$\pi^*(E\cap(\bigcup_{j=1}^n F_j)) = \sum_{j=1}^n \pi^*(E\cap F_j), \forall E\subseteq\Omega$$

proof: we will prove by mathematical induction

$n=1$: trivially true

let it be true for $n=k$ then

$$\pi^*(E\cap(\bigcup_{j=1}^k F_j)) = \sum_{j=1}^k \pi^*(E\cap F_j) \quad \forall E\in\Omega$$

$n=k+1$:

let $\{F_i\}_{i=1}^{k+1} \subseteq M$ be disjoint collection

since, $F_{k+1}\in M$, and $E\cap(\bigcup_{j=1}^{k+1} F_j)\in\Omega$

$$\pi^*(E\cap(\bigcup_{j=1}^{k+1} F_j)) = \pi^*(E\cap(\bigcup_{j=1}^{k+1} F_j)\cap F_{k+1}) + \pi^*(E\cap(\bigcup_{j=1}^{k+1} F_j)\cap F_{k+1}^c)$$

now as all $\{F_i\}_{i=1}^{k+1}$ are disjoint,

$$\left(\bigcup_{j=1}^{k+1} F_j\right)\cap F_{k+1} = F_{k+1}$$

$$\Rightarrow E\cap\left(\bigcup_{j=1}^{k+1} F_j\right)\cap F_{k+1} = E\cap F_{k+1}$$

and as all disjoint

$$F_{k+1}^c \supseteq \bigcup_{j=1}^{k+1} F_j$$

$$\Rightarrow\left(\bigcup_{j=1}^{k+1} F_j\right)\cap F_{k+1}^c = \bigcup_{j=1}^{k+1} F_j$$

$$\Rightarrow E\cap\left(\bigcup_{j=1}^{k+1} F_j\right)\cap F_{k+1}^c = E\cap\left(\bigcup_{j=1}^{k+1} F_j\right)$$

so, $\pi^*(E\cap(\bigcup_{j=1}^{k+1} F_j)\cap F_{k+1}) + \pi^*(E\cap(\bigcup_{j=1}^{k+1} F_j)\cap F_{k+1}^c)$

$= \pi^*(E\cap F_{k+1}) + \pi^*(E\cap(\bigcup_{j=1}^{k} F_j))$

$= \pi^*(E\cap F_{k+1}) + \sum_{i=1}^{k}\pi^*(E\cap F_i)$ ($\because$ induction hypothesis)

$\Rightarrow \pi^*(E\cap(\bigcup_{i=1}^{k+1} F_i)) = \sum_{i=1}^{k+1}\pi^*(E\cap F_i)$, so true for $n=k+1$ by induction. true overall

Exe: show that $\phi, \Omega \in M$, and $A \in M \Rightarrow A^c \in M$

Ans:

as $\pi^*(E) = \pi^*(E) \quad \forall E \subseteq \Omega$

$\Rightarrow \pi^*(E) = 0 + \pi^*(E)$

$\Rightarrow \pi^*(E) = \pi^*(\phi \cap E) + \pi^*(\Omega \cap E) \quad \forall E \subseteq \Omega$

$\Rightarrow \phi, \Omega \in M$ as $\phi^c = \Omega$ and $\Omega^c = \phi$

also trivial to see that if $A \in M$ i.e

$\pi^*(E) = \pi^*(A \cap E) + \pi^*(A^c \cap E) \quad \forall E \subseteq \Omega$

then $A^c \in M$

29th Jan:

we have seen π^* is additive on M, M is σ-algebra and $\mathcal{A} \Rightarrow \mathcal{F}_1(\mathcal{A}) \subseteq M$

we have also seen that $\pi^*|_{\mathcal{A}} = \nu$

Theorem: $\pi^*: M \longrightarrow [0, \infty]$ is a measure on M

Proof: all we have to show is that π^* is σ-additive

let $\{A_i\}_{i=1}^{\infty} \subseteq M$ be a disjoint collection then if $\pi^*(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \pi^*(A_i)$ then we are

but as π^* is outer measure, $\pi^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \pi^*(A_i)$ —①

now, $\bigcup_{i=1}^{\infty} A_i \supseteq \bigcup_{i=1}^{n} A_i \quad \forall n \in \mathbb{N}$

$\Rightarrow \pi^*(\bigcup_{i=1}^{\infty} A_i) \geq \pi^*(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \pi^*(A_i)$ ← disjoint union of sets

$(\because \pi^*$ is additive$)$

$\Rightarrow \pi^*(\bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} \pi^*(A_i)$ as $n \longrightarrow \infty$ —②

from ①, ② $\Rightarrow \pi^*$ is σ-additive

Note: from properties we have been seeing, the caratheodory extension's first part follows

$\mathcal{F}_1(\mathcal{A}) \subseteq M \subseteq \mathcal{P}(\Omega)$ ← in many cases containment is strict or equal

$\pi^*: M \longrightarrow [0,\infty]$ measure

$\Rightarrow \pi^*: \mathcal{F}_1(\mathcal{A}) \longrightarrow [0,\infty]$ measure

we have shown extension of $\nu: \mathcal{A} \longrightarrow [0,\infty]$ on $\mathcal{F}_1(\mathcal{A})$

$\pi^*|_{\mathcal{F}_1(\mathcal{A})}$ is a measure on $\mathcal{F}_1(\mathcal{A})$

$\pi^*|_{\mathcal{A}} = \nu$

Theorem: let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be an algebra and let $\mu_1, \mu_2 : \mathcal{F}_1(\mathcal{A}) \longrightarrow [0,\infty]$ be measures such that

$\mu_1(A) = \mu_2(A) \quad \forall A \in \mathcal{A}$

Suppose $\mu_1|_{\mathcal{A}} : \mathcal{A} \longrightarrow [0,\infty]$ is σ-finite

i.e $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t it is cover of Ω

$\Omega = \bigcup_{i=1}^{\infty} E_i$ and $\mu_1|_{\mathcal{A}}(E_i) < \infty \quad \forall i \in \mathbb{N}$

then $\mu_1(A) = \mu_2(A) \quad \forall A \in \mathcal{F}_1(\mathcal{A})$

Proof: wlog, we assume that $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ st $E_i \uparrow \Omega$ and $\mu_1(E_i) < \infty \quad \forall i \in \mathbb{N}$

as $F_1 = E_1$ and $F_2 = E_1 \cup E_2$, ... if not otherwise

let $F_i^\circ = E_i^\circ$ (wlog case) so, $\mu(F_i^\circ) \leq \sum_{j=1}^{i} \mu(E_j) < \infty$ as all $< \infty$

and $\bigcup_{K=1}^{\infty} F_K = \Omega$, $F_K \subseteq F_{K+1} \Rightarrow F_n \uparrow \Omega$
$\Rightarrow E_n \uparrow \Omega$

let $\mathcal{B}_n = \{E \in \mathcal{F}_1(\mathcal{A}) \mid \mu_1(E \cap E_n) = \mu_2(E \cap E_n)\}$ $\forall n \in \mathbb{N}$
then
$\mathcal{A} \subseteq \mathcal{B}_n$ $\forall n \in \mathbb{N}$ as $\forall A \in \mathcal{A} \Rightarrow A \cap E_n \in \mathcal{A}$
$\Rightarrow \mu_1(A \cap E_n) = \mu_2(A \cap E_n)$

now, $\mathcal{B}_n \subseteq \mathcal{F}_1(\mathcal{A})$ by construction, so if we show $\mathcal{B}_n$ is σ-algebra then we are done

so, it is sufficient to show that $\mathcal{B}_n$ is σ-algebra $\forall n \in \mathbb{N}$
as $\mathcal{A} \subseteq \mathcal{B}_n \Rightarrow \mathcal{F}_1(\mathcal{A}) \subseteq \mathcal{B}_n$

but $\mathcal{F}_1(\mathcal{A}) \supseteq \mathcal{B}_n \Rightarrow \mathcal{F}_1(\mathcal{A}) = \mathcal{B}_n$ $\forall n \in \mathbb{N}$

so, $\mu_1(E \cap E_n) = \mu_2(E \cap E_n)$ $\forall n \in \mathbb{N}$, $\forall E \in \mathcal{F}_1(\mathcal{A})$

$\Rightarrow \lim_{n \to \infty} \mu_1(E \cap E_n) = \lim_{n \to \infty} \mu_2(E \cap E_n)$

as μ_1, μ_2 are measures
they are σ-additive and so continuous

as $E_n \uparrow \Omega$, $E \cap E_n \uparrow E \Rightarrow \lim_{n \to \infty} \mu_1(E \cap E_n) = \mu_1(E)$
($\because \mu_1, \mu_2$ are continuous from below)

$\lim_{n \to \infty} \mu_2(E \cap E_n) = \mu_2(E)$

so for we have not used $\mu_1(E_i) < \infty$
so, $\mu_1(E) = \mu_2(E)$ $\forall E \subseteq \Omega$
if $\mathcal{B}_n$ is σ-algebra $\forall n \in \mathbb{N}$

now if $\mathcal{B}_n$ is a monotone-class then by monotone class theorem

$\mathcal{F}_1(\mathcal{A}) = \mu(\mathcal{A}) \subseteq \mu(\mathcal{B}_n) = \mathcal{B}_n$
($\mu(\mathcal{A})$: smallest monotone class by $\mathcal{A}$; $\mathcal{A} \subseteq \mathcal{B}_n$)

and so we are done

fix $n \in \mathbb{N}$, let $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{B}_n$ s.t $A_j \uparrow A$, we will show $A \in \mathcal{B}_n$

now as $A_j \in \mathcal{B}_n$, $\mu_1(A_j \cap E_n) = \mu_2(A_j \cap E_n)$ $\forall j \in \mathbb{N}$

$\lim_{j \to \infty} \mu_1(A_j \cap E_n) = \lim_{j \to \infty} \mu_2(A_j \cap E_n)$
($\because \mu_1, \mu_2$ continuous from below)

$\Rightarrow \mu_1(A \cap E_n) = \mu_2(A \cap E_n)$
$\Rightarrow A \in \mathcal{B}_n$

now let, $\{C_K\}_{K=1}^{\infty} \subseteq \mathcal{B}_n$ s.t $C_K \downarrow C$ then as $\mu_1(C_1 \cap E_n) < \infty$
& $\mu_2(C_1 \cap E_n) < \infty$

we have $\mu_1(C \cap E_n) = \lim_{K\to\infty} \mu_1(C_K \cap E_n)$

$= \lim_{K\to\infty} \mu_2(C_K \cap E_n)$

$= \mu_2(C \cap E_n)$

$\Rightarrow C \in \mathcal{B}_n$

Defn: let $\Omega \neq \phi$, let $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ be σ-algebra and let $\mu: \mathcal{F} \to [0,\infty]$ is a measure, then

(a) $(\Omega, \mathcal{F})$ is called a measurable space

(b) $(\Omega, \mathcal{F}, \mu)$ is called a measure space

Eg: let $\Omega \neq \phi$, define $C: \mathcal{P}(\Omega) \to [0,\infty]$ by

$C(A) =$ no of elements of A, $\forall A \in \mathcal{P}(\Omega)$

then C is a measure, then $(\mathbb{R}, \mathcal{B}, C)$ is a measure space
$\mathcal{B}$ is borel sigma field

Exe: show that counting measure C is a measure → done down

Note: Functionals on function space corresponds to a measure (finite dim is trivial, but infinite is non-trivial)

Eg: let D_1, D_2 be two countable dense subsets of $\mathbb{R}$ s.t $D_1 \cap D_2 = \phi$
($D_1 = \mathbb{Q}$, $D_2 = \sqrt{2} + \mathbb{Q}$)

let $\mu_1, \mu_2 : \mathcal{B} \to [0,\infty]$ by

$$\mu_1(A) = C(A \cap D_1)$$
$$\mu_2(A) = C(A \cap D_2) \quad \forall A \in \mathcal{B}$$

consider $\{\{x\} \mid x \in D_1\} \cup \{\mathbb{R} \setminus D_1\}$ this is countable and every set has finite μ_1-measure

$\Rightarrow \mu_1$ is σ-finite
similarly μ_2 is σ-finite, now $\mu_1 \neq \mu_2$ on $\mathcal{B}$
as $\mu_1(D_1) = \infty$
$\mu_2(D_1) = 0$

on $\mathcal{B}$ they don't agree, if we show on $\mathcal{A}$ agrees but this is not a violation as on $\mathcal{A}$, they are not σ-finite

Exe: for any $\phi \neq A \in \mathcal{A}(S)$ (S = semi open intervals), then $\mu_1(A) = \mu_2(A) = \infty$

but $\mu_1|_{\mathcal{A}(S)}$ is not σ-finite → done down

Exe: show that counting measure C is a measure

Ans:

$C: \mathcal{P}(\Omega) \to [0,\infty]$

$C(\phi) = 0$ ($\because \phi$ has no elements)

also for $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{P}(\Omega)$ s.t $E_i \cap E_j = \phi$ for $i \neq j$ $\quad E = \bigcup_{i=1}^{\infty} E_i$

case I: $\exists i \in \mathbb{N}$ s.t $C(E_i) = \infty$

in this case $\exists E_i$ with infinite many elements, so $\bigcup_{i=1}^{\infty} E_i \supseteq E_i$ will also have infinite elements

$\Rightarrow C(E) = \infty = \sum_{i=1}^{\infty} C(E_i)$

$(\because C(E_i) = \infty$ & $\sum_{i=1}^{\infty} C(E_i) = \infty)$

case II: $\forall i,\ C(E_i) < \infty$

in this case, all are finite, then

$\bigcup_{i=1}^{\infty} E_i$ = countable union of disjoint finite sets

then $\sum_{i=1}^{\infty} C(E_i)$ = sum of all elements of $\bigcup_{i=1}^{\infty} E_i$

$= C(E)$ as all are disjoint,
if ∞ then E also ∞
if finite, then E also finite

Exe: for any $\phi \neq A \in \mathcal{A}(S)$ (S = semi open intervals), then $\mu_1(A) = \mu_2(A) = \infty$

but $\mu_1|_{\mathcal{A}(S)}$ is not σ-finite

Soln: $\phi \neq A \in \mathcal{A}(S)$ S = semi-open intervals

$\mu_1(A) = C(A \cap D_1)$

$\mu_2(A) = C(A \cap D_2)$

now as $A = [a, b)$ for $a \neq b$

$A \cap D_1 = [a, b) \cap D_1$

$C([a,b) \cap D_1) = \infty$ ($\because D_1$ is countable & infinite many many in between)

similarly $C([a,b) \cap D_2) = \infty$

also trivial if $a = b$ then $C(\phi) = C(\phi)$

&, $\mu_1(A) = \mu_2(A)\ \forall A \in \mathcal{A}$

now, if $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t

$\mu_1(E_i) < \infty$ and $\Omega = \bigcup_{i=1}^{\infty} E_i$

then let $E_i = [a_i, b_i)$ $\exists i \in \mathbb{N}$ s.t $E_i \neq \phi$ as if all $E_i = \phi$ then $\bigcup E_i \neq \Omega$

but $\mu_1(E_i) = \infty$ this is a contradiction

&, μ_1 not σ-finite

2nd Feb:

Construction of Lebesgue measure:

In most of cases, measure is usually defined on semi-algebra / Algebra, and if its unique, then we can define it on σ-algebra

Hypothesis we will consider:

$F : \mathbb{R} \to \mathbb{R}$ that is
(i) non-decreasing
(ii) Right continious

$$F(a) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} F(a+\varepsilon)$$

F can be extended to $F: \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ via following definition

$$F(+\infty) = \sup \{F(x) \mid x \in \mathbb{R}\}$$

$$F(-\infty) = \inf \{F(x) \mid x \in \mathbb{R}\}$$

consider semi-algebra with

$$S = \{(a,b] \mid -\infty \le a \le b \le +\infty\} \setminus \{(-\infty, -\infty], (\infty, \infty]\}$$

with convention:

$$\phi = (a,a] \quad \forall a \in \mathbb{R}$$

Note: $\phi \neq (a,a]$ for $a = \pm\infty$

let, $\lambda_F : S \longrightarrow [0,\infty]$ by
$\lambda_F((a,b]) = F(b) - F(a)$

Lemma: let $\{(a_i, b_i] \mid 1 \le i \le k\}$ be disjoint collection in S with $(a_i, b_i] \subseteq (a,b]$ then

$$\sum_{i=1}^{k} \lambda_F((a_i,b_i]) \le \lambda_F((a,b])$$

Lemma: suppose $(a,b] \subseteq \bigcup_{i=1}^{k} (a_i, b_i]$ with $a_i < b_i$ $\forall i = 1, \ldots k$ and $a, b, a_i, b_i \in \mathbb{R}$

then

$$\sum_{i=1}^{k} F(b_i) - F(a_i) \ge F(b) - F(a)$$

5th Feb:

Theorem: Let $\{\alpha_{i,n}\} \subseteq [0,\infty)$ $\forall i,n \in \mathbb{N}$, suppose for each i, $\alpha_{i,n} \uparrow \alpha_i$ as $n \to \infty$ then

$$\sum_{i\in\mathbb{N}} \alpha_{i,n} \uparrow \sum_{i\in\mathbb{N}} \alpha_i \quad \text{as } n \to \infty, \uparrow \text{ is increasing}$$

Ex: Show above theorem → done done

Note: limit can be $+\infty$ also

Theorem: $\lambda_F : S \to [0,\infty]$ is a measure

proof:

$\lambda_F(\phi) = 0$, from definition

suppose $\{(a_i,b_i]\}_{i=1}^{\infty} \subseteq S$ be a disjoint collection s.t $\bigcup_{i=1}^{\infty}(a_i,b_i] = (a,b] \in S$

for all $k \in \mathbb{N}$, $\bigcup_{i=1}^{k}(a_i,b_i] \subseteq (a,b]$

$$\Rightarrow \sum_{i=1}^{k} \lambda_F((a_i,b_i]) \leq \lambda_F((a,b])$$

($\because$ lemma done in tutorial)

$$\Rightarrow \lim_{k\to\infty} \sum_{i=1}^{k} \lambda_F((a_i,b_i]) \leq \lambda_F((a,b])$$

$$\Rightarrow \sum_{i=1}^{\infty} \lambda_F((a_i,b_i]) \leq \lambda_F((a,b]) \quad \text{——} (1)$$

we need to now show, $\lambda_F((a,b]) \leq \sum_{i=1}^{\infty} \lambda_F((a_i,b_i])$

Case I: $a,b \in \mathbb{R}$

then, $\{a_i,b_i \mid i \in \mathbb{N}\} \subseteq \mathbb{R}$, fix $\varepsilon > 0$, then will show that

$$\lambda_F((a,b]) - \varepsilon \leq \sum_{i=1}^{\infty} \lambda_F((a_i,b_i]) + \varepsilon$$

By using right continuity of F at a and b_i, we can choose a', b_i' such that

$a < a' < b$ and $b_i < b_i'$ and $F(a') < F(a) + \varepsilon$ ($\because$ Right continuity)

and $F(b_i') < F(b_i) + \frac{\varepsilon}{2^i}$ $\forall i \in \mathbb{N}$

this gives $[a',b] \subseteq (a,b] \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i] \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i')$

by compactness of $[a',b]$, $\exists J$ s.t J is finite

$$[a',b] \subseteq \bigcup_{i\in J}(a_i,b_i')$$

$$\Rightarrow F(b) - F(a') \leq \sum_{i\in J} F(b_i) - F(a_i)$$

($\because$ lemma done in tutorial)

now, $\sum_{i=1}^{\infty} \lambda_F((a_i,b_i]) + \varepsilon = \sum_{i=1}^{\infty}\left(F(b_i) - F(a_i) + \frac{\varepsilon}{2^i}\right) > \sum_{i=1}^{\infty} F(b_i') - F(a_i)$

$> \sum_{i\in J} F(b_i') - F(a_i)$

$\geq F(b) - F(a')$

so, $\sum_{i=1}^{\infty} \lambda_F((a_i, b_i]) + \varepsilon > F(b) - F(a') \geq F(b) - F(a) - \varepsilon$

$\Rightarrow \sum_{i=1}^{\infty} \lambda_F((a_i, b_i]) + \varepsilon > F(b) - F(a) - \varepsilon$ as $\varepsilon \to 0$

$\Rightarrow \sum_{i=1}^{\infty} \lambda_F((a_i, b_i]) \geq F(b) - F(a)$ ——— ②

from ①, ② $\sum_{i=1}^{\infty} \lambda_F((a_i, b_i]) = F(b) - F(a)$

Case 2: $a \in \mathbb{R}$, $b = +\infty$

$\forall\, h \in \mathbb{N}$, consider

$$(a, h] = (a, b] \cap (-\infty, h]$$
$$= \bigcup_{i=1}^{\infty} ((a_i, b_i] \cap (-\infty, h])$$

then by case 1:

$$\sum_{i=1}^{\infty} \lambda_F((a_i, b_i] \cap (-\infty, h]) = \lambda_F((a, h])$$

we have to now show, $\lambda_F((c, d] \cap (-\infty, h]) \uparrow \lambda_F((c, d])$ as $h \to \infty$, $(c, d] \in S$

case (i): if $d \in \mathbb{R}$, then $\forall\, h > d$ we have result as

$$(c, d] \cap (-\infty, h] = (c, d] \text{ as } d < h$$

case (ii): if $d = \infty$, then

$$\lambda_F((c, d] \cap (-\infty, h]) = \lambda_F((c, h])$$
$$= F(h) - F(c)$$

then $\lim_{h\to\infty} \lambda_F((c, d] \cap (-\infty, h]) = \lim_{h\to\infty} (F(h) - F(c))$

$$= F(\infty) - F(c)$$
$$= \lambda_F((c, d]) \quad (\because d = \infty)$$

→ $\alpha_{i,n} \uparrow \alpha_i$ then $\sum \alpha_{i,n} \uparrow \sum \alpha_i$ as $n \to \infty$, $\alpha_{i,n} = \lambda_F((a_i, b_i] \cap (-\infty, n])$

so, from above theorem $\sum_{i=1}^{\infty} \lambda_F((a_i, b_i]) = \lambda_F((a, \infty])$

Exe: finish proof for $a = -\infty$ by taking seprate case 1, case 2 → love love

Note: observe that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} (n, n+1]$, countable disjoint union, and

$$\lambda_F((n, n+1]) < \infty \quad \forall\, n \in \mathbb{N}$$

so, λ_F is σ-finite, and $\forall\, F$, λ_F is unique

so, ① $\lambda_F : S \longrightarrow [0,\infty]$ is σ-finite measure

② $\lambda_F : \mathcal{A}(S) \longrightarrow [0,\infty]$ is σ-finite measure

③ $\pi_F^* =$ outer measure associated to λ_F, then

$\pi_F^* : \mathcal{M}_F \longrightarrow [0,\infty]$ is the unique carathéodory extension of λ_F

④ $(\mathbb{R}, \mathcal{B}, \pi_F^*)$ is a σ-finite measure space

⑤ $(\mathbb{R}, \mathcal{M}_F, \pi_F^*)$ is a σ-finite measure space

Defn: λ_F is called Lebesgue-Stieltjes measure

In particular, $F(x) = x$, then $F(\pm\infty) = \pm\infty$

$\lambda((a,b]) = b - a$ (here λ_F is denoted by λ)

and $\mathcal{M} = \mathcal{L}$, $\pi_{id}^* = \lambda$ (extension also denoted by λ (∵ unique))

Defn: $\lambda : \mathcal{L} \longrightarrow [0,\infty]$ is called the Lebesgue measure on $\mathbb{R}$, and $\mathcal{L}$ is called the Lebesgue sigma algebra on $\mathbb{R}$, and the elements of $\mathcal{L}$ are called Lebesgue measurable sets

Exe: show above theorem

Ans:

$\{\alpha_{i,n}\} \subseteq [0,\infty)$ $\forall i,n \in \mathbb{N}$ for each i, i.e. $\alpha_{i,n} \uparrow \alpha_i$ as $n \longrightarrow \infty$

and $\alpha_{i,n} \le \alpha_{i,n+1}$ $\forall n \in \mathbb{N}$

and $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n > N$ $|\alpha_{i,n} - \alpha_i| < \varepsilon$

then as $\alpha_{p,n} \le \alpha_{i,n+1}$ $\forall p, \forall n \in \mathbb{N}$

$\Rightarrow \sum_{i\in\mathbb{N}} \alpha_{i,n} \le \sum_{i\in\mathbb{N}} \alpha_{i,n+1}$ $\forall n \in \mathbb{N}$

and as $\forall \varepsilon/2^p > 0$, $\exists N_p \in \mathbb{N}$ s.t. $|\alpha_{p,n} - \alpha_i| < \frac{\varepsilon}{2^p}$ $\forall n > N_p$

then, $\left|\sum_{i=1}^{K} \alpha_{p,n} - \sum_{i=1}^{K} \alpha_p\right| \le \sum_{i=1}^{K} |\alpha_{i,n} - \alpha_i| < \sum_{i=1}^{K} \frac{\varepsilon}{2^p}$

$\forall n > \max\{N_1, N_2, \ldots N_K\} = \tilde{N}_K$

$\Rightarrow \lim_{K\to\infty} \left|\sum_{i=1}^{K} \alpha_{i,n} - \sum_{i=1}^{K} \alpha_i\right| \le \lim_{K\to\infty} \sum_{i=1}^{K} |\alpha_{i,n} - \alpha_i| < \lim_{K\to\infty} \sum_{i=1}^{K} \frac{\varepsilon}{2^i}$

$\forall n > \lim_{K\to\infty} \tilde{N}_K$

so, $\left|\sum_{i\in\mathbb{N}} \alpha_{i,n} - \sum_{i\in\mathbb{N}} \alpha_i\right| < \varepsilon$ $\forall n > \max\{N_1, N_2, \ldots\}$

or $\sum_{i\in\mathbb{N}} \alpha_{i,n} \uparrow \sum_{i\in\mathbb{N}} \alpha_i$

Exe: finish proof for $a = -\infty$ by taking separate case 1, case 2

Ans: now, $(n, b] = (a, b] \cap (n, \infty]$

$$= \bigcup_{i=1}^{\infty} (]a_i, b_i] \cap]n, \infty])$$

then, by case 1 if $b \in \mathbb{R}$

$$\sum_{i=1}^{\infty} \lambda_F(]a_i, b_i] \cap]n, \infty]) = \lambda_F(]n, b])$$

now, we have to show $\lambda_F(]c, d] \cap]n, \infty]) \uparrow \lambda_F(]c, d])$

if $c \in \mathbb{R}$, then

$\forall\, n > c$

$]c, d] \cap]n, \infty] =]c, d]$

if $c = -\infty$ then

$$\lambda_F(]c, d] \cap]n, \infty]) = \lambda_F(]n, d]) = \lambda_F(d) - \lambda_F(n)$$

$$\lim_{n \to -\infty} \lambda_F(]c, d] \cap]n, \infty]) = \lambda_F(d) - \lambda_F(c)$$

so, $\sum_{i=1}^{\infty} \lambda_F(]a_i, b_i] \cap]n, \infty]) \uparrow \sum_{i=1}^{\infty} \lambda_F(]a_i, b_i]) = \lambda_F(]-\infty, b])$ $(\because n \to -\infty)$

if $b = \infty$, then, by case 2:

$$\sum_{i=1}^{\infty} \lambda_F(]a_i, b_i] \cap]n, \infty]) = \lambda_F(]n, \infty])$$

now, if we show $\lambda_F(]c, d] \cap]n, \infty]) = \lambda_F(]n, \infty])$ we are done

it follows similarly from previous one

so, we are done

9th Feb:

Completion of measure:

In carathéodory extension, we showed we can extend λ_F uniquely to larger σ-algebra, and we also saw
for λ_{id} or λ
$\mathcal{B} \subseteq \mathcal{L}$

Suppose $A \in \mathcal{F}$ and $\mu(A)=0$, let $E \subseteq A$, then we want to know if $E \in \mathcal{F}$ (↳ null set wrt measure)
i.e we want to know if they are measurable or not

Let $\mathcal{F}_1 = \mathcal{B}$ $A \in \mathcal{B}$ s.t $\lambda(A)=0$
or we want to know if $\forall E \subseteq A \in \mathcal{B}$ s.t $\lambda(A)=0$ then $E \in \mathcal{B}$ ← this is not true

this is not true, but
$\forall E \subseteq A \in \mathcal{L}$ s.t $\lambda(A)=0$ then $E \in \mathcal{L}$ ← this is true

Note: $\mathcal{B}$ is not complete but $\mathcal{L}$ is complete

Defn: Let $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ be σ-algebra and let $\mu: \mathcal{F} \to [0,\infty]$ be a measure then $\mathcal{F}$ is said to be complete wrt μ if $\forall E \subseteq A$ with $A \in \mathcal{F}$ and $\mu(A)=0$ implies $E \in \mathcal{F}$
then $(\Omega, \mathcal{F}, \mu)$ is said to be complete measure space
so completion is a property of the measure, so $\mathcal{B}, \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$ only one is complete wrt λ

Defn: Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space, then the completion of $(\Omega, \mathcal{F}, \mu)$ is a measure space $(\Omega, \mathcal{F}_0, \mu_0)$ with the following properties:

① $\mathcal{F}_0$ is complete wrt μ_0

② $\mathcal{F} \subseteq \mathcal{F}_0$ and $\mu_0|_{\mathcal{F}} = \mu$

③ if $(\Omega, \mathcal{F}_1, \mu_1)$ is a complete measure space satisfying $\mathcal{F} \subseteq \mathcal{F}_1$ and $\mu_1|_{\mathcal{F}} = \mu$ then
$\mathcal{F}_0 \subseteq \mathcal{F}_1$ and $\mu_1|_{\mathcal{F}_0} = \mu_0$

Let $\mathcal{F}_\mu = \{A \cup N \mid A \in \mathcal{F}$ and $N \subseteq F$ where $F \in \mathcal{F}$ with $\mu(F)=0\}$

Exe: Show that $\mathcal{F}_\mu$ is a σ-algebra → done down

also, trivial to see $\mathcal{F} \subseteq \mathcal{F}_\mu$ as if $N=\emptyset$ then this follows

now, let $\bar{\mu}: \mathcal{F}_\mu \longrightarrow [0,\infty]$ by $\bar{\mu}(A \cup N) = \mu(A)$
$A \in \mathcal{F}$, $N \subseteq F$ s.t $F \in \mathcal{F}$ and $\mu(F)=0$

Exe: Show that $\bar{\mu}$ is a measure → done down

Exe: Show that $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is a complete measure space → done down

Exe: Show that $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is the completion of $(\Omega, \mathcal{F}, \mu)$ → done down

Recall the hypothesis of Carathéodory Extension thm:
$\nu: \mathcal{A} \to [0,\infty]$ σ-finite measure

$\pi^*: M \to [0,\infty]$ measure

and $\mathcal{F}_1(\mathcal{A}) \subseteq M$

now, what we want to show this for $\mathbb{R}, \mathcal{B}, \mathcal{L}$

Lemma: Given any $A \in M$, $\exists B \in \mathcal{F}_1(\mathcal{A})$ s.t $A \subseteq B$ with $\pi^*(A) = \pi^*(B)$

proof:

If $\pi^*(A) = \infty$, then (trivial) choose $B = \Omega$ and so we will get the equality

if $\pi^*(A) < \infty$, then for given $n \in \mathbb{N}$, $\exists$ cover $A \subseteq \bigcup_{j=1}^{\infty} E_{n,j}$ s.t

$\{E_{n,j}\}_{j=1}^{\infty} \subseteq \mathcal{A}$ s.t

$$\pi^*(A) \leq \sum_{j=1}^{\infty} \nu(E_{n,j}) \leq \pi^*(A) + \frac{1}{n}$$

now, let $B_n = \bigcup_{j=1}^{\infty} E_{n,j}$ and define $B = \bigcap_{n=1}^{\infty} B_n$

so, then $A \subseteq B$ as $A \subseteq B_n \ \forall n \in \mathbb{N}$

$$\Rightarrow A \subseteq \bigcap_{n=1}^{\infty} B_n = B$$

clearly, $B \in \mathcal{F}_1(A)$ ($\because$ definition of $\mathcal{F}_1(\mathcal{A})$, as $E_{n,j} \in \mathcal{F}_1(\mathcal{A}), \forall i,n$)

so, $\pi^*(A) \leq \pi^*(B)$ — ① ($\because$ monotonicity)

now, $\forall n \in \mathbb{N}$, $\pi^*(B) \leq \pi^*(B_n) \leq \sum_{j=1}^{\infty} \pi^*(E_{n,j})$ ($\because$ subadditivity of π^*)

$$= \sum_{j=1}^{\infty} \nu(E_{n,j}) \quad (\because E_{n,j} \in \mathcal{A})$$

$$\leq \pi^*(A) + \frac{1}{n}$$

as this is true for all $n \in \mathbb{N} \Rightarrow \pi^*(B) \leq \pi^*(A)$ — ②

so, $\pi^*(B) = \pi^*(A)$ ($\because$ ①, ②)

theorem: M is the completion of $\mathcal{F}_1(\mathcal{A})$ with respect to measure $\pi^*|_M$. In particular, M_F is completion of $\mathcal{B}(\mathbb{R})$ wrt λ_F

proof: we have to show that M is completion wrt $\pi^*|_M$

let $A \subseteq B$, $B \in M$ with $\pi^*(B) = 0$

then $A \in M$ if we show

$$\pi^*(E) \geq \pi^*(A \cap E) + \pi^*(A^c \cap E) \quad \forall E \subseteq \Omega$$

now, $\pi^*(A^c \cap E) \leq \pi^*(E)$ ($\because$ monotonicity)

or, other way

$$\pi^*(E \cap A) \leq \pi^*(B) = 0 \Rightarrow \pi^*(E \cap A) = 0$$

so, $\pi^*(E) \geq \pi^*(A^c \cap E) + 0$ ←

$\Rightarrow A \in M$, and M is complete

for completion, we have to show that for completion

$\forall A \in M$, $\exists B \in \mathcal{H}(\mathcal{A})$ and $N \subseteq H$ s.t $H \in \mathcal{H}(A)$ with $\pi^*(H)=0$

s.t $A = B \cup N$

($\pi^*(A)=\infty$ case is trivial as $\exists B \in \mathcal{H}(\mathcal{A}) = \Omega$, $N = \phi$)

now, if $A \in M$, with $\pi^*(A) < \infty$, then $\exists B \in \mathcal{H}(\mathcal{A})$ s.t $A \subseteq B$ and

$\pi^*(A) = \pi^*(B)$ ($\because$ prev lemma)

then $B = A \cup (B \setminus A) \Rightarrow \pi^*(B \setminus A) = 0$

as $\pi^*(A) < \infty$

$\Rightarrow \pi^*(B) < \infty$

so, $\pi^*(B \setminus A) = 0$

now, suppose $A \in M$, then By σ-finiteness

we have $A = \bigcup_{n=1}^{\infty} A_n$, s.t $\pi^*(A_n) < \infty \ \forall n \in \mathbb{N}$

$\{A_n\} \subseteq M$ ($\because \exists \{E_i\}_{i=1}^{\infty}$ s.t $\bigcup_{i=1}^{\infty} E_i = \Omega$, $A_i = A \cap E_i$, $\mu(E_i) < \infty$)

therefore $\exists \{B_n\}_{n=1}^{\infty} \subseteq \mathcal{H}(\mathcal{A})$ s.t $A_n \subseteq B_n$ and $\pi^*(A_n) = \pi^*(B_n) \ \forall n \in \mathbb{N}$

also, $\pi^*(B_n \setminus A_n) = 0$, let $B = \bigcup_{n=1}^{\infty} B_n$

$$B = A \cup \left(\bigcup_{n=1}^{\infty} (B_n \setminus A_n)\right)$$

by subadditivity, $\pi^*\left(\bigcup_{n=1}^{\infty} (B_n \setminus A_n)\right) = 0$ as $\pi^*(B_p \setminus A_p) = 0 \ \forall p \in \mathbb{N}$

then $B \setminus A \subseteq \bigcup_{n=1}^{\infty} (B_n \setminus A_n)$

$\Rightarrow \pi^*(B \setminus A) = 0$

so, observe that $A^c = \Omega \setminus A$

$= (\Omega \setminus B) \cup (B \setminus A)$

with $\Omega \setminus B \in \mathcal{H}(\mathcal{A})$

and $B \setminus A \subseteq \bigcup_{n=1}^{\infty} (B_n \setminus A_n) \in \mathcal{H}(\mathcal{A})$

$\pi^*\left(\bigcup_{n=1}^{\infty} (B_n \setminus A_n)\right) = 0$

so $A^c \in M \Rightarrow A \in M$ ($\because M$ is σ-algebra)

Exe: Show that $\mathcal{H}_\mu$ is a σ-algebra

Ans:

$$\mathcal{H}_\mu = \{A \cup N \mid A \in \mathcal{H},\ N \subseteq F \text{ s.t } F \in \mathcal{H},\ \mu(F) = 0\}$$

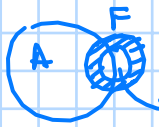
then ① $A = \phi$, $N = \phi$ $F = \phi \Rightarrow \phi \in \mathcal{H}_\mu$

$A = \Omega$, $N = \phi$ $F = \phi \Rightarrow \Omega \in \mathcal{H}_\mu$

② $\forall B \in \mathcal{H}_\mu \Rightarrow B = A \cup N$

$N \subseteq F$ s.t $\mu(F) = 0$, $F \in \mathcal{H}$

$A^c \in \mathcal{H}$ as $A \in \mathcal{H}$



$N \cup (F \setminus N) = F$

$N^c \cap (F \setminus N)^c = F^c$

$\Rightarrow (N^c \cap (F \setminus N)^c)^c \cup (F \setminus N) = F \cup (F \setminus N)$

$\Rightarrow N^c = F^c \cup (F \setminus N)$

then $B^c = A^c \cap N^c = (A^c \cap F^c) \cup (A^c \cap (F \setminus N))$

as $F \setminus N \subseteq F \Rightarrow A^c \cap (F \setminus N) \subseteq F$

also, $A^c \cap F^c \in \mathcal{H}$ as $A \in \mathcal{H}$, $F \in \mathcal{H}$

$\Rightarrow B^c \in \mathcal{H}_\mu$

③ for $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_\mu$

$\forall i \in \mathbb{N}$, we have $\exists A_i \in \mathcal{H}$, $N_i \subseteq F_i$ s.t $F_i \in \mathcal{H}$, $\mu(F_i) = 0$

s.t $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A_i \cup N_i)$

$= (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} N_i)$

then $\exists A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{H}$ ($\because \forall i \in \mathbb{N}, A_i \in \mathcal{H}$)

and $\exists F = \bigcup_{i=1}^{\infty} F_i \in \mathcal{H}$ by property of σ-algebra $N_i \subseteq F_i$

s.t $\mu(F) \leq \sum_{i=1}^{\infty} \mu(F_i) = 0$ ($\because$ subadditivity) $\Rightarrow \mu(F) = 0$

$\exists N = \bigcup_{i=1}^{\infty} N_i \subseteq F = \bigcup_{i=1}^{\infty} F_i$

s.t $B = A \cup N \in \mathcal{H}_\mu$

so, $\mathcal{H}_\mu$ is σ-algebra

exe: show that $\bar{\mu}$ is a measure

Ans:

let $B \in \mathcal{H}_\mu$ then $\exists A \in \mathcal{H}$, $N \subseteq F \in \mathcal{H}$ s.t $\mu(F) = 0$

$\bar{\mu}(A \cup N) = \mu(A)$

① $A \cup N = \phi$ then $A = \phi$, $N = \phi$

$\Rightarrow \bar{\mu}(\phi) = \mu(\phi) = 0$

② $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{H}_\mu$ s.t $\bigcup_{i=1}^{\infty} E_i \in \mathcal{H}_\mu$ and for $i \neq j$ $E_i \cap E_j = \phi$

then $\exists A_i \in \mathcal{H}$, $N_i \subseteq F_i \in \mathcal{H}$ s.t $\mu(F_i) = 0$ $\forall i \in \mathbb{N}$

and from above exercise

$\bigcup_{i=1}^{\infty} E_i = (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} N_i)$ where $\bigcup_{i=1}^{\infty} A_i \in \mathcal{H}$, $\bigcup_{i=1}^{\infty} N_i \subseteq \bigcup_{i=1}^{\infty} F_i \in \mathcal{H}$, $\mu(\bigcup_{i=1}^{\infty} F_i) = 0$

$\Rightarrow \bar{\mu}(\bigcup_{i=1}^{\infty} E_i) = \bar{\mu}((\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} N_i)) = \mu(\bigcup_{i=1}^{\infty} A_i)$

as $\forall i \neq j$ we have $E_i \cap E_j = \phi$

as, $A_i \subseteq E_i$, $A_j \subseteq E_j$

$\Rightarrow A_i \cap A_j \subseteq E_i \cap E_j$

$\Rightarrow A_i \cap A_j \subseteq \phi$

$\Rightarrow A_i \cap A_j = \phi$

$\Rightarrow \mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$

$\Rightarrow \bar{\mu}(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \bar{\mu}(A_i \cup N_i) = \sum_{i=1}^{\infty} \bar{\mu}(E_i)$

so, $\bar{\mu}$ is a measure

Exe: show that $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is a complete measure space

Ans:

$(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is complete measure space

as $\mathcal{F}_\mu$ is σ-algebra

$\bar{\mu}: \mathcal{F}_\mu \to [0, \infty]$ is measure (∵ previous exercises)

for $\forall E$ s.t $E \subseteq B \in \mathcal{F}_\mu$

$$\bar{\mu}(B) = 0$$

then $\exists A \in \mathcal{F}, N \subseteq F \in \mathcal{F}$ s.t $\mu(F) = 0$

$$\bar{\mu}(B) = \mu(A) = 0$$

$$E \subseteq B \Rightarrow E \subseteq A \cup N \subseteq A \cup F$$

as $A \in \mathcal{F}, F \in \mathcal{F}$

$$\mu(A \cup F) \leq \mu(A) + \mu(F) \leq 0 \quad (\because \text{subadditivity})$$

$$\Rightarrow \mu(A \cup E) = 0$$

so, $E = \phi \cup E$ s.t $\phi \in \mathcal{F}$, $\exists A \cup F \in \mathcal{F}$ s.t $E \subseteq A \cup F$, $\mu(A \cup F) = \phi$

$$\Rightarrow E \in \mathcal{F}_\mu$$

so, $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is complete measure space

Exe: show that $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is the completion of $(\Omega, \mathcal{F}, \mu)$

Ans:

① $(\Omega, \mathcal{F}_\mu, \bar{\mu})$ is complete is shown

② $\forall A \in \mathcal{F} \Rightarrow A = A \cup \phi$, $\phi \subseteq \phi \in \mathcal{F}$ s.t $\mu(\phi) = 0$

$\Rightarrow A \in \mathcal{F}_\mu$

so, $\mathcal{F} \subseteq \mathcal{F}_\mu$

also, $\forall A \in \mathcal{F}$, $\bar{\mu}(A) = \bar{\mu}(A \cup \phi) = \mu(A) \Rightarrow \bar{\mu}|_{\mathcal{F}} = \mu$

③ now, $(\Omega, \mathcal{F}_1, \mu_1)$ is a complete measure space, $\mathcal{F} \subseteq \mathcal{F}_1$, $\mu_1|_{\mathcal{F}} = \mu$

then $\forall E \subseteq F$ s.t $F \in \mathcal{F}$, $\mu(F) = 0$ we get $F \in \mathcal{F} \subseteq \mathcal{F}_1 \Rightarrow F \in \mathcal{F}_1$ and $\bar{\mu}(F) = \mu(F) = 0$

or $E \in \mathcal{F}_1$ as $\mathcal{F}_1$ is complete

then $\forall A \in \mathcal{F}$, $\forall E \subseteq F$ s.t $F \in \mathcal{F}$, $\mu(F) = 0$

$\Rightarrow A \cup E \in \mathcal{F}_1$ as $A \in \mathcal{F}$, $E \in \mathcal{F}_1$

$\Rightarrow \mathcal{F}_\mu \subseteq \mathcal{F}_1$

also, $\forall A \cup E = B \in \mathcal{F}_\mu$ as $A, E \in \mathcal{F}_1$

$$\mu_1(A \cup E) = \mu_1(A \cup (E \setminus A))$$

$$= \mu_1(A) + \mu_1(E \setminus A)$$

$E \setminus A \subseteq E \subseteq F \in \mathcal{F}$ s.t $\mu(F) = 0 \Rightarrow \mu_1(F) = 0$

$\Rightarrow \mu_1(E \setminus A) = 0$ (∵ monotonicity)

and so, $\mu_1(A \cup E) = \mu_1(A) = \mu(A) = \bar{\mu}(A \cup E) \Rightarrow \mu_1|_{\mathcal{F}_\mu} = \bar{\mu}$

11th Feb:

Quiz-1 - 16th Feb monday in class (till last class)

Measurable fuctions:

consider the extended borel σ-algebra on $\overline{\mathbb{R}}$

$$\overline{\mathcal{B}(\mathbb{R})} = \{A \cup B \mid A \in \mathcal{B}(\mathbb{R}),\ B \subseteq \{-\infty, \infty\}\}$$

Defn: let $(\Omega, \mathcal{F})$ be a measurable space, A fuction $f: \Omega \to \overline{\mathbb{R}}$ is said to be measurable if

$$f^{-1}(B) \in \mathcal{F} \quad \forall\ B \in \overline{\mathcal{B}(\mathbb{R})}$$

Note: we dont use $\mathcal{L}$ but $\mathcal{B}(\mathbb{R})$ as we can approximate measurable fuctions in $\mathcal{L}$ as $\mathcal{B}(\mathbb{R})$ (Borel measurable fuctions)

Note: Measurable fuctions does not use measure

Lemma: let $(\Omega, \mathcal{F})$ be a measurable space, let $f: \Omega \to \overline{\mathbb{R}}$ be a fuction then the following are equivalent:

1. f is measurable
2. $f^{-1}([-\infty, x]) \in \mathcal{F}\ \forall\ x \in \mathbb{R}$
3. $f^{-1}([-\infty, x)) \in \mathcal{F}\ \forall\ x \in \mathbb{R}$
4. $f^{-1}((x, +\infty]) \in \mathcal{F}\ \forall x \in \mathbb{R}$
5. $f^{-1}([x, +\infty]) \in \mathcal{F}\ \forall\ x \in \mathbb{R}$

proof: suppose f is measrable, then

$(1 \Rightarrow 2)$ $1 \Rightarrow 2$ is trivial as, $[-\infty, x] \in \overline{\mathcal{B}(\mathbb{R})}\ \forall x \in \mathbb{R}$

$\Rightarrow f^{-1}([-\infty, x]) \in \mathcal{F}$ ($\because f$ is measrable)

$(1 \Leftarrow 2)$ to show this, consider

$$\mathcal{C} = \{A \in \overline{\mathcal{B}(\mathbb{R})} \mid f^{-1}(A) \in \mathcal{F}\}$$

if $\mathcal{C} = \overline{\mathcal{B}(\mathbb{R})}$ we are done

now, by (2): $[-\infty, x] \in \mathcal{C}\ \forall\ x \in \mathbb{R}$

now if $\mathcal{C}$ is a σ-algebra, then

$$\mathcal{C} \subseteq \overline{\mathcal{B}(\mathbb{R})} \Rightarrow \mathcal{F}(\{[-\infty, x] \mid x \in \mathbb{R}\}) \subseteq \mathcal{C} \subseteq \overline{\mathcal{B}(\mathbb{R})}$$

$$\overline{\mathcal{B}(\mathbb{R})} \quad (\because \text{Exercise below})$$

$$\Rightarrow \mathcal{C} = \overline{\mathcal{B}(\mathbb{R})}$$

So, it is sufficient to show that $\mathcal{C}$ is σ-algebra this follows from the fact that

f^{-1} respects set theoritic operations

i.e if $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{C}$ then $\bigcup_{i=1}^{\infty} f^{-1}(A_i) \in \mathcal{F}_1 \Rightarrow f^{-1}(\bigcup_{i=1}^{\infty} A_i) \in \mathcal{F}_1 \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$

similarly, $(f^{-1}(A))^c = f^{-1}(A^c)$ ↑ Property of σ-algebra

Exe: $\mathcal{H}(\{[-\infty, x] \mid x \in \mathbb{R}\}) = \overline{\mathcal{B}(\mathbb{R})}$ → done down

Exe: show ① ⇔ ③ ⇔ ④ ⇔ ⑤ for above theorem → done down

Defn: Let $(\Omega, \mathcal{F}_1)$ be a measurable space, a function $f: \Omega \to \mathbb{R}$ is said to be a basic function if $\exists$ a finite disjoint collection

$$\{E_k \mid k = 1, 2, \ldots n\} \subseteq \mathcal{F}_1$$

and

$$\{c_k\}_{k=1}^{n} \subset \mathbb{R} \text{ s.t } f = \sum_{k=1}^{n} c_k \chi_{E_k}$$

Let's say we have given

$$\{c_k\}_{k=1}^{n} \subset \mathbb{R} \quad \{E_k \mid k = 1, 2 \ldots n\} \subseteq \mathcal{F}_1$$

to extend this to $\Omega = (\bigcup_{k=1}^{n} E_k) \sqcup B$

then

$$f = \sum_{k=1}^{n} c_k \chi_{E_k} + 0 \cdot \chi_B$$

where $\{E_k \mid k = 1 \ldots n\} \cup \{B\}$ is a partition of Ω

Exe: show that a basic measurable function $s = \{E_k \mid k = 1, \ldots n\} \cup \{B\}$ is measurable

Ans:

Let $A \in \overline{\mathcal{B}(\mathbb{R})}$, then

$$f^{-1}(A) = \bigcup_{\{k \mid c_k \in A\}} E_k$$

↑ at most finite

$\Rightarrow f^{-1}(A)$ measurable

So, every basic function is measurable

Note: we cannot directly do integration as that will lead to problem $\infty - \infty$

if $c_1 > 0$, $c_2 < 0$ $\mu(E_1) = \infty$ $\mu(E_2) = \infty$

So, now let $(\Omega, \mathcal{F}_1, \mu)$ be a measure space and let $f: \Omega \to [0, \infty)$ be a basic function, then

$$\exists \{E_k \mid k = 1, \ldots n\} \subseteq \mathcal{F}_1 \text{ partition (disjoint) of } \Omega$$

and $\{c_k\}_{k=1}^{n} \subset [0, \infty)$ s.t $f = \sum_{k=1}^{n} c_k \chi_{E_k}$

Defn: $\int_{\Omega} f \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k)$

one more issue we run into is that $\{c_k\}$ $\{E_k\}$ is not unique, so now we have to see if it is well defined

lemma: $\sum_{k=1}^{n} c_k \mu(E_k)$ does not depend on the choice of the representation of f

proof: Suppose $f = \sum_{k=1}^{n} c_k \chi_{E_k}$ and $f = \sum_{l=1}^{m} d_l \chi_{F_l}$

then we have to show

$$\sum_{k=1}^{n} c_k \mu(E_k) = \sum_{l=1}^{m} d_l \mu(F_l)$$

Suppose, $\exists\, k_0 \in \{1,2,\dots n\}$ and $l_0 \in \{1,2,\dots m\}$ s.t $E_{k_0} \cap F_{l_0} \neq \phi$

then $\exists\, \omega \in E_{k_0} \cap F_{l_0}$ and

so, $f(\omega) = c_{k_0} = d_{l_0}$

Observe that $\mu(E_k) = \mu(E_k \cap \Omega)$

$$= \mu\left(E_k \cap \left(\bigcup_{l=1}^{m} F_l\right)\right)$$

$$= \mu\left(\bigcup_{l=1}^{m} (E_k \cap F_l)\right)$$

↑ disjoint

$$= \sum_{l=1}^{m} \mu(E_k \cap F_l)$$

so, $\sum_{k=1}^{n} c_k \mu(E_k) = \sum_{k=1}^{n}\sum_{l=1}^{m} c_k \mu(E_k \cap F_l)$

$c_k \mu(E_k \cap F_l)$ → if ϕ then 0 else $c_k = d_l$

$$= \sum_{k=1}^{n}\sum_{l=1}^{m} d_l \mu(E_k \cap F_l)$$

↓ if 0 then d_l does not matter

$$= \sum_{l=1}^{m}\sum_{k=1}^{n} d_l \mu(E_k \cap F_l)$$

$$= \sum_{l=1}^{m} d_l \sum_{k=1}^{n} \mu(E_k \cap F_l)$$

$\sum_{k=1}^{n}$ ↑ all disjoint

$$= \sum_{l=1}^{m} d_l \mu\left(\bigcup_{k=1}^{n} E_k \cap F_l\right)$$

$$= \sum_{l=1}^{m} d_l \mu(F_l)$$

Exe: $\mathcal{A}(\{[-\infty, x] \mid x \in \mathbb{R}\}) = \overline{\mathcal{B}(\mathbb{R})}$

Ans:

as $[-\infty, x] = (-\infty, x] \cup \{-\infty\}$

as $(-\infty, x] \in \mathcal{B}(\mathbb{R})$, $\{-\infty\} \subseteq \{-\infty, \infty\}$

$\Rightarrow (-\infty, x] \cup \{-\infty\} \in \overline{\mathcal{B}(\mathbb{R})}$

$\Rightarrow \{[-\infty, x] \mid x \in \mathbb{R}\} \subseteq \overline{\mathcal{B}(\mathbb{R})}$

$\Rightarrow \mathcal{F}_1(\{[-\infty, x] \mid x \in \mathbb{R}\}) \subseteq \overline{\mathcal{B}(\mathbb{R})}$ —— ①

now let $A = (a, b)$ $a, b \in \mathbb{R}$
$B \subseteq \{-\infty, \infty\}$

then $(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$

$= \bigcup_{n=1}^{\infty} \Big(\underbrace{[-\infty, b - \frac{1}{n}]}_{\in \mathcal{F}_1(S)} \setminus \underbrace{[-\infty, a]}_{\in \mathcal{F}_1(S)} \Big)$

$\in \mathcal{F}_1(S)$

now, $\{\infty\} = \bar{\mathbb{R}} \setminus [-\infty, \infty)$

$= \Omega \setminus [-\infty, x]$ $\quad x \to \infty, x \in \mathbb{R}$

$[-\infty, x]$ form $\in \mathcal{F}_1(S)$

so, $\{\infty\} \in \mathcal{F}_1(S)$

and then $\{\infty\}^c = [-\infty, \infty) \in \mathcal{F}_1(S)$ as $(-\infty, \infty) = (a, b)$ form $\in \mathcal{F}_1(S)$

$\Rightarrow [-\infty, \infty) \setminus (-\infty, \infty) = \{-\infty\} \in \mathcal{F}_1(S)$

so, $A \cup B \in \mathcal{F}_1(S)$ $\forall A, B$

$\Rightarrow \mathcal{F}_1(\{A \cup B\}) \subseteq \mathcal{F}_1(S)$

$\Rightarrow \overline{\mathcal{B}(\mathbb{R})} \subseteq \mathcal{F}_1(\{[-\infty, x] \mid x \in \mathbb{R}\})$ —— ②

so, from ① , ② $\overline{\mathcal{B}(\mathbb{R})} = \mathcal{F}_1(\{[-\infty, x] \mid x \in \mathbb{R}\})$

Ex: show ① ⇔ ③ ⇔ ④ ⇔ ⑤ for above theorem

Ans:

① ⇒ ③, ④, ⑤ is trivial (∵ same reason as ②)

let $\mathcal{C} = \{A \in \overline{\mathcal{B}(\mathbb{R})} \mid f^{-1}(A) \in \mathcal{F}\}$ $\mathcal{C}$ is σ-algebra from before

then in ③: if we construct $\mathcal{C}$

then $\{[-\infty, x] \mid x \in \mathbb{R}\} \subseteq \mathcal{C} \subseteq \overline{\mathcal{B}(\mathbb{R})}$

similar to ②, $\mathcal{F}_1(\{[-\infty, x] \mid x \in \mathbb{R}\}) = \overline{\mathcal{B}(\mathbb{R})} \subseteq \mathcal{C} \subseteq \overline{\mathcal{B}(\mathbb{R})}$

(∵ similar calculation as above)

and so we get the equivalence, same for ④, ⑤

12th Feb:

Lebesgue integration:

theorm: let $(\Omega, \mathcal{F})$ be a measurable space, and let $f: \Omega \to [0,\infty]$ be measurable function, then there exists a sequence of non-negative basic functions $\{f_n\}_{n=1}^{\infty}$ s.t

1) $\lim_{n\to\infty} f_n(\omega) = f(\omega) \quad \forall \omega \in \Omega$
2) $\forall \omega \in \Omega, f_{n+1}(\omega) \geq f_n(\omega)$ (i.e $f_n \uparrow f$)
3) If f is bounded, then $f_n \to f$ uniformly

proof: fix $n \in \mathbb{N}$, consider the following finite partion

$[0,\infty]$

$n=1$: $[0,1) \cup [1,\infty]$ (0, ∞) $\quad n\cdot 2^{n-1} = (1)(2^1)-1 = 1$

$n=2$: $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 2, \infty$ $\quad n\cdot 2^{n}-1 = (2)(2^2)-1 = 7$

$[0,\frac{1}{4}) \cup [\frac{1}{4},\frac{2}{4}) \cdots [\frac{7}{4},\frac{8}{4}) \cup [2,\infty]$

general form: $[0,\infty] = \bigcup_{k=0}^{n\cdot 2^{n-1}} \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \cup [n,\infty]$

↑ this will not make any difference, we can use 2, 3, ...

this produces a finite partion of Ω:

$$\Omega = \bigcup_{k=0}^{n\cdot 2^{n-1}} f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \cup f^{-1}\left([n,\infty]\right)$$

(all are disjoint (∵ property of f^{-1}))

let $f_n : \Omega \to [0,\infty]$ by

$$f_n = \sum_{k=0}^{n\cdot 2^{n-1}} \frac{k}{2^n} \chi_{f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)} + n\,\chi_{f^{-1}([n,\infty])}$$

↓ non-negative basic function

observe that $f_n(\omega) \leq f(\omega) \quad \forall \omega \in \Omega$ (∵ trivial)

suppose, $f(\omega) = \infty$, then $f_n(\omega) = n \quad \forall n \in \mathbb{N}$

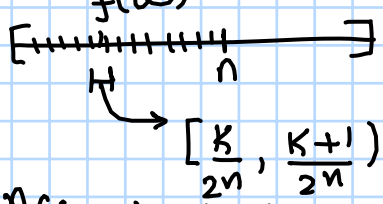
$\Rightarrow \lim_{n\to\infty} f_n(\omega) = \infty = f(\omega)$

suppose $f(\omega) < \infty$ for some $\omega \in \Omega$
then $\exists\, n_0 \in \mathbb{N}$ s.t

$f(\omega) < n_0$ $\quad 0 \quad f(\omega) \quad n_0 \quad \infty$

fix $n \geq n_0$, then $\exists\ k \in \mathbb{N}$ s.t and $0 \leq k \leq n \cdot 2^n - 1$

$$\frac{k}{2^n} \leq f(\omega) < \frac{k+1}{2^n}$$



$\Rightarrow k \leq 2^n f(\omega) < k+1$

$\Rightarrow k = \lfloor 2^n f(\omega) \rfloor$ ← property of floor function

then $f_n(\omega) = \dfrac{\lfloor 2^n f(\omega) \rfloor}{2^n}$

$\Rightarrow f_n(\omega) \geq \dfrac{2^n f(\omega) - 1}{2^n}$ ← one more property of floor function

also, $f_n(\omega) \leq \dfrac{2^n f(\omega) + 1}{2^n}$

$\Rightarrow \dfrac{2^n f(\omega) - 1}{2^n} \leq f_n(\omega) \leq \dfrac{2^n f(\omega) + 1}{2^n}$

$\Rightarrow \lim_{n\to\infty} \left(\dfrac{2^n f(\omega) - 1}{2^n}\right) \leq \lim_{n\to\infty} f_n(\omega) \leq \lim_{n\to\infty} \dfrac{2^n f(\omega) + 1}{2^n}$

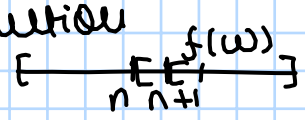
$\Rightarrow f(\omega) \leq \lim_{n\to\infty} f_n(\omega) \leq f(\omega)$

$\Rightarrow f(\omega) = \lim_{n\to\infty} f_n(\omega)$

$\therefore \lim_{n\to\infty} f_n(\omega) = f(\omega) \quad \forall\ \omega \in \Omega$

now, fix $n \in \mathbb{N}$, then suppose

case I: $\omega \in \Omega$ s.t $f(\omega) \geq n+1$ then by construction $f_{n+1}(\omega) = n+1 > n = f_n(\omega)$

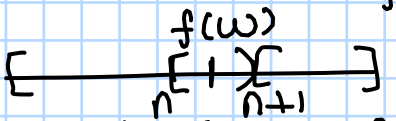


case II: $\omega \in \Omega$ s.t $f(\omega) < n+1$ then $n \leq f(\omega)$

$$f_{n+1}(\omega) = \frac{\lfloor 2^{n+1} f(\omega) \rfloor}{2^{n+1}} \quad (\because \text{property})$$

$f_n(\omega) = n \Rightarrow \dfrac{\lfloor 2^{n+1} f(\omega) \rfloor}{2^{n+1}} \geq \dfrac{2^{n+1}}{2^{n+1}} f(\omega) \geq \dfrac{2^{n+1}(n)}{2^{n+1}}$

$\Rightarrow f_{n+1}(\omega) \geq f_n(\omega)$



case III: $\omega \in \Omega$ s.t $\exists\ k$ s.t $0 \leq k \leq n \cdot 2^n - 1$, then $f(\omega) < n$,

$f(\omega) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ $\quad f_n(\omega) = \frac{k}{2^n}$

now, $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) = \left[\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right) = \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right) \cup \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right)$

($\left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right) = I_1$, $\left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right) = I_2$)

if $f(\omega) \in I_1$, then $f_{n+1}(\omega) = \frac{2k}{2^{n+1}} = \frac{k}{2^n} = f_n(\omega)$

if $f(\omega) \in I_2$, then $f_{n+1}(\omega) = \frac{2k+1}{2^{n+1}} = \frac{k}{2^n} + \frac{1}{2^{n+1}} > \frac{k}{2^n} = f_n(\omega)$

$\Rightarrow f_{n+1}(\omega) \geq f_n(\omega)$

so, from 3 cases, $f_{n+1}(\omega) \geq f_n(\omega)$ $\forall \omega \in \Omega$

now, if $\exists\, N \in \mathbb{N}$ s.t $f: \Omega \to [0, N)$ then $\forall n \geq N$

$$0 \leq f(\omega) - f_n(\omega) \leq \frac{1}{2^n} \quad \forall \omega \in \Omega$$

as $f_n(\omega) = \frac{[2^n f(\omega)]}{2^n}$

$$0 \leq f_n(\omega) - f(\omega) \leq \frac{2^n f(\omega) + 1 - f(\omega)}{2^n}$$

so, $f_n \to f$ uniformly if f is bounded

def^n: suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f: \Omega \to [0, \infty]$ is a measurable function, then

$\int_\Omega f\, d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu$ where $\{f_n\}_{n=1}^\infty$ is an increasing seq of non-negative basic functions, s.t $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ $\forall \omega \in \Omega$

now, to proceed further, we have to show that it's well defined

Lemma: suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and $\{f_n\}^\infty$ is an increasing sequence of non-negative basic functions and $g: \Omega \to [0, \infty]$ is a non-negative basic function on Ω

if $g(\omega) \leq \lim_{n\to\infty} f_n(\omega)$ $\forall \omega \in \Omega$, then $\int_\Omega g\, d\mu \leq \lim_{n\to\infty} \int_\Omega f_n d\mu$

proof: now $g: \Omega \to [0, \infty]$ is a non-negative basic function

so, $\exists\, \{E_i\}_{i=1}^n \subseteq \mathcal{F}$ disjoint s.t $\bigcup_{i=1}^n E_i = \Omega$ and $\{c_i\}_{i=1}^n \subseteq [0, \infty]$

s.t $g = \sum_{i=1}^n c_i \chi_{E_i}$

now, $\int_\Omega g\, d\mu = \sum_{i=1}^n c_i \mu(E_i)$ ($\because$ by definition)

and $\exists \{F_{i,j}\}_{i=1}^{n_j} \subseteq \mathcal{F}$ disjoint s.t $\bigcup_{i=1}^{n_j} F_{i,j} = \Omega$

$\exists \{d_{i,j}\}_{i=1}^{n_j} \subseteq [0,\infty]$ s.t $f_j = \sum_{i=1}^{n_j} d_{i,j} \chi_{F_{i,j}}$

now, $\int_\Omega F_j \, d\mu = \sum_{i=1}^{n_j} d_{i,j} \mu(F_{i,j})$

as $g(\omega) \leq \lim_{j\to\infty} F_j(\omega) \quad \forall \omega \in \Omega$

$\Rightarrow \sum_{i=1}^{n} c_i \chi_{E_i}(\omega) \leq \lim_{j\to\infty} \sum_{i=1}^{n_j} d_{i,j} \chi_{F_{i,j}}(\omega) \quad \forall \omega \in \Omega$

lets usage $\{F_{i,j}\}_{i=1}^{n_j}$ and make it more finer

$\{E_k \cap F_{i,j}\}_{i=1,k=1}^{n_j,n}$ now trivial to see this is disjoint, also as we have seen welldefindness $\int_\Omega f_n \, d\mu$ value will not change

also, $E_k = \bigcup_{i=1}^{n_j} (E_k \cap F_{i,j})$ → all disjoint

$\Rightarrow \forall \omega \in E_k \cap F_{i,j} \neq \phi$ ($\because$ if ϕ then trivial)

$g(\omega) = c_k \leq \lim_{j\to\infty} d_{i,j}$

$\Rightarrow c_k \mu(E_k \cap F_{i,j}) \leq \lim_{j\to\infty} d_{i,j} \mu(E_k \cap F_{i,j})$

($\because$ multiplying both sides by positive quantity)

$\Rightarrow \sum_{i=1}^{n_j} c_k \mu(E_k \cap F_{i,j}) \leq \lim_{j\to\infty} \sum_{i=1}^{n_j} d_{i,j} \mu(E_k \cap F_{i,j})$

$\Rightarrow c_k \mu(E_k) \leq \lim_{j\to\infty} \sum_{i=1}^{n_j} d_{i,j} \mu(E_k \cap F_{i,j})$ ← everything is positive so limit interchange

$\Rightarrow \sum_{k=1}^{n} c_k \mu(E_k) \leq \lim_{j\to\infty} \sum_{k=1}^{n} \sum_{i=1}^{n_j} d_{i,j} \mu(E_k \cap F_{i,j})$

$\Rightarrow \sum_{k=1}^{n} c_k \mu(E_k) \leq \lim_{j\to\infty} \sum_{i=1}^{n_j} d_{i,j} \mu(F_{i,j})$

$\Rightarrow \sum_{k=1}^{n} c_k \mu(E_k) \leq \lim_{j\to\infty} \int_\Omega f_j \, d\mu$

theorm: let $(\Omega, \mathcal{F}, \mu)$ be a measure space, suppose $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are inc sequences of non-negative basic functions on Ω s.t

$f_n(\omega) \to f(\omega) \quad \forall \omega \in \Omega \quad \text{as } n \to \infty$

and $g_n(\omega) \to f(\omega) \quad \forall \omega \in \Omega \quad \text{as } n \to \infty$

where $f: \Omega \to [0, \infty]$ is a measurable function

then, $\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \lim_{n\to\infty} \int_\Omega g_n \, d\mu$

proof: by hypothesis, $\forall K \in \mathbb{N}, \; g_K(\omega) \leq \lim_{n\to\infty} f_n(\omega) \quad \forall \omega \in \Omega$

$= f(\omega) \quad (\because g_K \uparrow f)$

then from above lemma

$$\int_\Omega g_K \, d\mu \leq \lim_{n\to\infty} \int_\Omega f_n \, d\mu$$

$$\Rightarrow \lim_{K\to\infty} \int_\Omega g_K \, d\mu \leq \lim_{n\to\infty} \int_\Omega f_n \, d\mu$$

similarly, $\lim_{K\to\infty} \int_\Omega f_K \, d\mu \leq \lim_{n\to\infty} \int_\Omega g_n \, d\mu$

$$\Rightarrow \lim_{n\to\infty} \int_\Omega f_n \, d\mu = \lim_{n\to\infty} \int_\Omega g_n \, d\mu$$

19th Feb:

defn: let $(\Omega, \mathcal{F}, \mu)$ be a meas. space and let $f: \Omega \to \bar{\mathbb{R}}$ be a meas function

1) if f is non-negative then f is said to be lebesgue integrable if

$$\int_\Omega f \, d\mu < \infty$$

2) otherwise f is said to be lebesgue integrable if $\int_\Omega f^+ \, d\mu < \infty$ and $\int_\Omega f^- d\mu < \infty$ then $\int_\Omega f \, d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$

properties of lebesgue integral

lemma: suppose $f, g: \Omega \to [0, \infty]$ are measurable functions and let $\alpha, \beta \in \mathbb{R}_{\geq 0}$ then

1) $\int_\Omega (\alpha f + \beta g) \, d\mu = \alpha \int_\Omega f \, d\mu + \beta \int_\Omega g \, d\mu$

2) if $0 \leq g \leq f$ then $\int_\Omega g \, d\mu \leq \int_\Omega f \, d\mu$

proof: 1) let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be increasing seq of non-negative basic functions s.t $f_n(\omega) \uparrow f(\omega)$ and $g_n(\omega) \uparrow g(\omega)$ $\forall \omega \in \Omega$ as $n \to \infty$

then $\{\alpha f_n + \beta g_n\}_{n=1}^\infty$ is an increasing sequence of non-negative basic functions, with $(\alpha f_n + \beta g_n)(\omega) \uparrow (\alpha f + \beta g)(\omega)$ $\forall \omega \in \Omega$ as $n \to \infty$

$$\Rightarrow \int_\Omega \alpha f + \beta g \, d\mu = \lim_{n \to \infty} \int_\Omega \alpha f_n + \beta g_n \, d\mu$$

$$= \alpha \lim_{n \to \infty} \int_\Omega f_n \, d\mu + \beta \lim_{n \to \infty} \int_\Omega g_n \, d\mu$$

$$= \alpha \int_\Omega f \, d\mu + \beta \int_\Omega g \, d\mu$$

2) first we will consider basic functions which approximate g

$\forall n \in \mathbb{N}$

$$g_n \leq g \leq f = \lim_{n \to \infty} f_n$$

$$\Rightarrow g_n \leq \lim_{n \to \infty} f_n$$

$$\Rightarrow \int_\Omega g_n \, d\mu \leq \lim_{n \to \infty} \int_\Omega f_n \, d\mu \quad (\because \text{lemma also})$$

$$\Rightarrow \int_\Omega g_n \, d\mu \leq \int_\Omega f \, d\mu$$

$$\Rightarrow \lim_{n \to \infty} \int_\Omega g_n \, d\mu = \int_\Omega g \, d\mu \leq \int_\Omega f \, d\mu$$

Note: from above lemma, if $\int_\Omega f \, d\mu = \infty$ or $\int_\Omega g \, d\mu = \infty$ then

$$\int_\Omega (\alpha f + \beta g) \, d\mu = \infty$$

if $\{f_n\}$ are measurable s.t $f_n \to f$, then as $\underline{\lim} f_n, \overline{\lim} f_n$ are measurable we get $\underline{\lim} f_n = \overline{\lim} f_n \Rightarrow f$ is measurable ($\because$ done in tutorial)

theorem: (Monotone convg theorem) suppose $f_n : \Omega \to [0, \infty]$ is an increasing sequence of measurable functions s.t $f_n(\omega) \uparrow f(\omega)$ $\forall \omega \in \Omega$ as $n \to \infty$

then $\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu$

proof: for each $n \in \mathbb{N}$, choose a sequence $\{f_{n,k}\}_{k=1}^{\infty}$ of non-negative basic functions s.t $f_{n,k}(\omega) \uparrow f_n(\omega)$ $\forall \omega \in \Omega$, now define

$$g_k(\omega) = \max_{n \le k} \{f_{n,k}(\omega)\} \quad \forall k \in \mathbb{N}$$

← we are varying n, not k

observe that $\{g_k\}_{k=1}^{\infty}$ is a seq of non-negative basic functions

also, $f_{n,k}(\omega) \le f_{n,k+1}(\omega)$ $\forall \omega \in \Omega$ ← $f_{n,k}(\omega) \le \max(f_{n,k}(\omega))$ & $\max(f_{n,k}) \le \max_{n+1 \le k} f_{n,k}$

$\Rightarrow \{g_k\}_{k=1}^{\infty}$ is a pointwise increasing seq of non-negative basic functions

let $g(\omega) = \lim_{k\to\infty} g_k(\omega)$ $\forall \omega \in \Omega$

for $n \le k$:

$f_{n,k} \le f_n \le f_k$ pointwise

$\Rightarrow \max_{n \le k} \{f_{n,k}\} \le \max_{n \le k} \{f_k\} = f_k$

$\Rightarrow g_k \le f_k$ pointwise

for $n \le k$, $f_{n,k} \le g_k \le f_k \le f$ pointwise

now let $k \to \infty$:

$\Rightarrow f_n \le g \le f$ pointwise

$n \to \infty$

$\Rightarrow f \le g \le f$ pointwise

$\Rightarrow f = g$

from $f_{n,k} \le g_k \le f_k \le f \Rightarrow \int_\Omega f_{n,k} \, d\mu \le \int_\Omega g_k \, d\mu \le \int_\Omega f_k \, d\mu$ ($\because$ lemma above)

now for $n \le k$

$$\lim_{k\to\infty} \int_\Omega f_{n,k} \, d\mu \le \lim_{k\to\infty} \int_\Omega g_k \, d\mu \le \lim_{k\to\infty} \int_\Omega f_k \, d\mu$$

$$\Rightarrow \int_\Omega f_n \, d\mu \le \int_\Omega f \, d\mu \le \lim_{k\to\infty} \int_\Omega f_k \, d\mu$$

$$\Rightarrow \lim_{k\to\infty} \int_\Omega f_k \, d\mu \le \int_\Omega f \, d\mu \le \lim_{k\to\infty} \int_\Omega f_k \, d\mu \quad (\because f = g)$$

so, $\lim_{k\to\infty} \int_\Omega f_k \, d\mu = \int_\Omega f \, d\mu$

theorem: (Fatou's lemma) Suppose $f_n : \Omega \to [0, \infty]$ is a sequence of measurable functions then $\int_\Omega \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu$

proof: for $n \in \mathbb{N}$ let $g_n = \inf_{k \geq n} f_k$ then g_n is an increasing sequence of non-negative measurable functions

$$\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n = g$$

then using the fact that $g_n \leq f_n \; \forall \, n \in \mathbb{N}$

$$\Rightarrow \int_\Omega g_n \, d\mu \leq \int_\Omega f_n \, d\mu \quad (\because \text{prev theorem})$$

$$\Rightarrow \lim_{n\to\infty} \int_\Omega g_n \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu$$

$$\Rightarrow \lim_{n\to\infty} \int_\Omega g_n \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu$$

$$\left(\because \lim_{n\to\infty} \int_\Omega g_n \, d\mu = \int_\Omega g \, d\mu \text{ by MCT i.e } \overline{\lim} = \underline{\lim} = \lim\right)$$

$$\Rightarrow \int_\Omega g \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu$$

$$\Rightarrow \int_\Omega \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n \, d\mu$$